Department of Mathematics University of Houston

MATH 6361

Applicable Analysis

Spring 2022

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 1, due Thursday, January 27, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Establish the identity

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\|z-\frac{1}{2}(x+y)\|^2$$

for x, y, z in a real or complex inner product space.

Problem 2

Let P be a bounded linear map on a real or complex Hilbert space H. Show that if $P^2 = P$ (meaning $P(Px) = Px$ for each $x \in H$) and for each $x, y \in H$, $\langle Px, y\rangle = \langle x, Py\rangle$, then the range of P is closed and P is the orthogonal projection onto its range. Hint: If $Px = 0$ then $x \perp \mathrm{ran}(P)$.

Problem 3

Let K be a closed, convex set in a real Hilbert space H and $x_0 \in H \setminus K$. Recall that there is a unique $y_0 \in K$ such that $\|y_0 - x_0\| = \inf_{y\in K} \|y - x_0\|$. Show that $z \in K$ satisfies $z = y_0$ if and only if for each $y \in K$, $\langle x_0 - z, y\rangle \le \langle x_0 - z, z\rangle$. Hint: Consider for $t \in [0, 1]$ the squared distance between x_0 and $ty_0 + (1-t)z$.

Problem 4

The complex exponentials $u_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, define an orthonormal basis for $L^2([-\pi, \pi])$, so for $f \in L^2([-\pi, \pi])$, $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\langle f, u_n\rangle|^2$. Evaluate both sides of this identity for (a) $f(x) = 1$, if $x \ge 0$ and $f(x) = 0$, if $x < 0$ and (b) $f(x) = \pi - |x|$. On one side, you get a number, on the other, a series.