MATH 6361
Applicable Analysis
Spring 2022

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 2, due Thursday, February 3, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Explain why $C([0,1])$, equipped with the L^p-norm for $1 \leq p < \infty$ and $p \neq 2$ cannot become an inner product space for which the norm is induced by the inner product. Hint: Consider two functions f and g, $f(x) = 1$ and $g(x) = 1$ if $0 \leq x < 1/2$ and $g(x) = -1$ if $1/2 \leq x \leq 1$.

Problem 2

Let H be a Hilbert space and F a bounded linear functional on H. Show that there exists a unique $z \in H$ such that $Fx = \langle x, z \rangle$ and $\|F\| = \|z\|$, where $\|F\|$ is the operator norm of F. Hint: The kernel of F is a closed subspace. If $F \neq 0$, consider $z' = x - Px$ where P is the projection onto the kernel of F and x satisfies $Fx \neq 0$.

Problem 3

Find

$$\min\left\{ \int_{-1}^{1} |x^2 - a - bx|^2 dx : a, b \in \mathbb{R} \right\}.$$

Hint: This minimum can be viewed as the minimum squared distance between the function $f(x) = x^2$ and elements in the subspace spanned by $\{v_1, v_2\}$ with $v_1(x) = 1$ and $v_2(x) = x$ in $L^2([-1,1])$.

Problem 4

Find the function $f \in C([-\pi, \pi])$, which satisfies

$$\int_{-\pi}^{\pi} f(x) x dx = 1 \text{ and } \int_{-\pi}^{\pi} f(x) \sin(x) dx = 2$$

and has minimal L^2-norm. Show that this function is unique.