MATH 6361

Applicable Analysis

Spring 2022

First name: ____________________ **Last name:** ________________________ **Points:**

Assignment 3, due Thursday, February 10, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

By (our) definition of $L^2([-\pi, \pi])$ as metric completion, continuous functions are dense in it. Consider the orthonormal basis $\{u_n\}_{n\in\mathbb{Z}}$ given by $u_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ for $L^2([-\pi, \pi])$. Use the properties of this orthonormal basis to prove that any $f \in L^2([-\pi, \pi])$ can be approximated arbitrarily closely by a continuous, periodic function, meaning for any $\epsilon > 0$, there is $g \in C([-\pi, \pi])$ with $\|f - g\| < \epsilon$ and $g(-\pi) = g(\pi)$.

Problem 2

Show that the Fourier series for the function $f(x) = x^2$ on $[-\pi, \pi]$ is given by

$$f(x) = \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(e^{inx} + e^{-inx})$$

and explain why this equality holds (as stated) pointwise for each $x \in [-\pi, \pi]$. Choose a suitable x to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

Problem 3

For $n \in \{0, 1, 2, \dots\}$, define the normalized Legendre polynomials by $p_0 = \frac{1}{\sqrt{2}}$ and $p_n(x) = (\frac{2n+1}{2^{2n+1}(n!)^2})^{1/2} \frac{d^n}{dx^n}(x^2 - 1)^n$. Show that $\{p_n\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2([-1, 1])$. Hint: Show that this is an orthonormal system (use integration by parts!). To prove that linear combinations are dense, you can appeal to the Weierstrass approximation theorem. You may use without proof that

$$\int_{-1}^{1} (x^2 - 1)^n dx = \frac{(-1)^n (n+1) 4^{n+1} (n!)^2}{(2(n+1))!} \, .$$

Problem 4

Prove that if $\frac{\alpha}{2\pi}$ is irrational, then for any 2π-periodic continuous function f on $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(k\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \, .$$

Hint: First show this for the special case $f(x) = e^{imx}$, $m \in \mathbb{Z}$, then use that the right-hand side of this identity defines a bounded linear functional on $f \in L^2([-\pi, \pi])$.