MATH 6361

Applicable Analysis

Spring 2022

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 4, due Thursday, February 17, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that weak convergence of a sequence in the closed unit ball of a separable Hilbert space is equivalent to convergence with respect to a metric

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} |\langle x - y, f_j \rangle|$$

where $(f_j)_{j=1}^{\infty}$ is a suitable sequence of vectors.

Problem 2

Let H be a Hilbert space. Show that the sequence $(x_n)_{n=1}^{\infty}$ converges weakly to x_0 in H if and only if for each fixed $k \in \mathbb{N}$, $\langle x_n, x_k \rangle \to \langle x_0, x_k \rangle$. Hint: Let $Y = \{y \in H : \langle x_n, y \rangle \to \langle x_0, y \rangle\}$ then this is a subspace. Why is Y closed? Apply the orthogonal projection onto Y to convert between weak convergence in H and weak convergence in Y.

Problem 3

Prove that a bounded sequence $(x^{(n)})_{n=1}^{\infty}$ in ℓ^2 is weakly convergent to $x \in \ell^2$ if and only if for each $k \in \mathbb{N}$, the sequence of k-th entries $(x_k^{(n)})_{n=1}^{\infty}$ converges to x_k. Hint: Use that any bounded sequence has a subsequence that is weakly convergent.

Problem 4

Let A be a closed, bounded and convex set in a Hilbert space H. Show that there is a vector $x^* \in A$ that has the smallest norm among all vectors in A.