MATH 6361
Applicable Analysis
Spring 2022

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 5, due Thursday, March 3, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let T be the operator on ℓ^2 defined by $(Tx)_k = x_k/k$. Show that if $(x^{(n)})_{n=1}^\infty$ is a sequence in ℓ^2 that converges weakly to 0, then $Tx^{(n)} \to 0$ in norm. Hint: Quote known results about such a sequence $(x^{(n)})_{n=1}^\infty$ in ℓ^2.

Problem 2

Define a simple real Sobolev-type space

$$H = \left\{ f \in C([-1,1]) : f(x) = \int_{[-1,x]} g dt \text{ for } x \in [-1,1], g \in L^2([-1,1]), f(1) = 0 \right\}$$

equipped with the inner product

$$\langle f, g \rangle = \int_{[-1,1]} f' g' dt$$

and the corresponding norm. Show that the linear functional $F : H \to \mathbb{R}$ defined by

$$F(f) = f(0)$$

is bounded. What is the unique function $h \in H$ guaranteed by the Riesz representation theorem satisfying for each $f \in H$ that $F(f) = \langle f, h \rangle$? Hint: Fundamental theorem of calculus.

Problem 3

Let $\epsilon > 0$ and

$$T_\epsilon = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}$$

define an operator on the Hilbert space $\mathbb{C}^2$, equipped with the standard inner product $\langle x, y \rangle = x_1 \overline{y_1} + x_2 \overline{y_2}$. Find the operator norm of T_ϵ and compare with $\max_{\|x\| \le 1} |\langle T_\epsilon x, x \rangle|$. Find T_ϵ^{-1}, compute its operator norm and compare with $\delta = \min_{\|x\|=1} |\langle T_\epsilon x, x \rangle|$.

Problem 4

The numerical range of an operator T on a Hilbert space H is the set $\rho(T) = \{\langle Tx, x \rangle, \|x\| = 1\}$. If T is bounded, linear and self-adjoint (meaning for $x, y \in H$, $\langle Tx, y \rangle = \overline{\langle Ty, x \rangle}$) and $\rho_1 = \langle Tx, x \rangle < \rho_2 = \langle Ty, y \rangle$ with $\|x\| = \|y\| = 1$, then show $[\rho_1, \rho_2] \subset \rho(T)$. Hint: Consider the values of the quadratic form associated with T restricted to $(1-t)x + ty$, $t \in [0,1]$.