MATH 6361

Applicable Analysis

Spring 2022

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 6, due Thursday, March 24, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that if A is a bounded, self-adjoint linear operator on a complex Hilbert space H and I the identity operator, then $T_\alpha = I + i\alpha A$ with $\alpha \in \mathbb{R}$ has a bounded inverse with norm $\|T_\alpha^{-1}\| \le 1$.

Problem 2

Let T be the operator on $L^2([-1,1])$ defined by

$$Tf(x) = \int_{[-1,1]} (x-y)f(y)dy\,.$$

Show that T is a Hilbert-Schmidt operator and compute its Hilbert-Schmidt norm $\|T\|_{HS}$. Hint: Consider an orthonormal basis of polynomials $\{1/\sqrt{2}, x/\sqrt{2/3}, \dots\}$.

Problem 3

Recall that an operator is of rank m if its range is m-dimensional. Show that any Hilbert-Schmidt operator T on ℓ^2 can be approximated in operator norm by a finite-rank operator, so for any $\epsilon > 0$, there is $m \in \mathbb{N}$ and an operator S of rank m on ℓ^2 such that $\|S - T\| < \epsilon$. Hint: Consider the canonical basis $(e_n)_{n=1}^\infty$ and define T_m for $m \in \mathbb{N}$ by

$$T_m x = \sum_{n=1}^{m} e_n \langle Tx, e_n \rangle\,.$$

Problem 4

If T is a bounded linear operator on a Hilbert space, prove that

$$\|TT^*\| = \|T^*T\| = \|T\|^2\,.$$