MATH 6361

Applicable Analysis

Spring 2022

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 7, due Thursday, March 31, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Consider the operator T defined on $f \in C([0,1])$ by

$$Tf(x) = xf(x)$$

and continuously extended to all f in $L^2([0,1])$. Prove that T is self-adjoint, but not compact.

Problem 2

Consider the Hilbert space ℓ^2 with the canonical orthonormal basis $(e_n)_{n=1}^{\infty}$ and the linear operator T satisfying

$$Te_k = \frac{1}{k} e_{k+1}$$

for each $k \in \mathbb{N}$. Show that T is compact, but that it has no eigenvectors.

Problem 3

Show that the operator V on $L^2([0,1])$ defined by

$$Vf(x) = \int_0^x f(t)dt$$

is compact. Hint: Recall the compactness of equicontinuous, closed and bounded families in $C([0,1])$ with respect to the sup-norm.

Problem 4

Let $K(s,t) = 4\cos(s-t)$ and T the extension of

$$Tf(t) = \int_{[-\pi,\pi]} K(t,s)f(s)ds$$

from $f \in C([-\pi,\pi])$ to all of the complex Hilbert space $L^2([-\pi,\pi])$. Find all the eigenvalues and a basis of eigenvectors for T. Hint: Fourier series.