Department of Mathematics University of Houston

MATH 6361
Applicable Analysis
Spring 2022

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 8, due Thursday, April 7, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let H be a complex Hilbert space. Use the spectral theorem to show that if T is self-adjoint and compact, then for any $\epsilon > 0$, there is a finite-rank operator S such that $\|T - S\| < \epsilon$. Hint: If $D = T - S$ is self-adjoint and compact, and $\{\mu_n\}$ are the eigenvalues of D, then $\|D\| \le \sup_n |\mu_n|$.

Problem 2

Show that if T is a compact operator on a complex, infinite-dimensional Hilbert space H and $\langle Tx, x\rangle \ge 0$ for each $x \in H$, then it is self-adjoint and has only non-negative eigenvalues $\lambda_n \ge 0$. With the corresponding orthonormal basis of eigenvectors $(x_n)_{n=1}^\infty$, define $Sy = \sum_{j=1}^\infty \sqrt{\lambda_n}\langle y, x_n\rangle x_n$. Show that S is a well-defined operator on H that is self-adjoint, compact and also has only non-negative eigenvalues, and satisfies $S^2 = T$.

Problem 3

Show that if an operator U on a complex Hilbert space is unitary, so $UU^* = U^*U = I$, and $U - I$ is compact, then U has a basis of eigenvectors $\{u_n\}$ with corresponding eigenvalues that satisfy $\lambda_n \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Problem 4

Show that if T is normal and compact on H, then it has an eigenvalue $\lambda \in \mathbb{C}$ such that $\|T\| = |\lambda|$. Hint: If T has eigenvalue λ, then T^*T has eigenvalue $|\lambda|^2$.