Department of Mathematics University of Houston

MATH 6361

Applied Analysis
Spring 2022

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 9, due Thursday, April 28, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let H be a real Hilbert space and K be a non-empty, convex, closed and bounded set and $x \notin K$. Show that there exists a bounded linear functional f such that $\inf_{y \in K} f(y) > f(x)$. Hint: First treat the special case $x = 0$. Recall that there is an element in K which minimizes the norm.

Problem 2

Consider ℓ^2 as a real Hilbert space, containing each square-summable sequence $x = (x_1, x_2, \dots)$. Consider the sets

$$A = \{x \in \ell^2 : k|x_k - k^{-2/3}| \leq x_1 \text{ for each } k \in \mathbb{N}\}$$

and

$$B = \{x \in \ell^2 : x_k = 0 \text{ if } k \geq 2\}\,.$$

a. Prove that A and B are closed convex sets and that $A \cap B = \emptyset$.

b. Show that $A - B = \{x \in \ell^2 :$ there is $C \geq 0$ such that $k|x_k - k^{-2/3}| \leq C$ for each $k \geq 2\}$.

c. Use the preceding result to show that $A - B$ is dense in ℓ^2.

d. Prove that A and B cannot be separated by a bounded linear functional.

Problem 3

Let $\{x_1, x_2, \dots, x_n\} \subset \mathbb{R}$, and let $\epsilon = \min\{|x_i - x_j| : i \neq j\}$. Suppose that there is a function $F : \mathbb{R} \to \{+1, -1\}$ that is onto. For $\lambda > 0$ and each $j \in \{1, 2, \dots, n\}$, define $f_j \in L^2(\mathbb{R})$ by

$$f_j(y) = \frac{1}{2\lambda} e^{-|y - x_j|/\lambda}\,.$$

Consider for each x_j the half-open interval $I_j = [x_j - \epsilon/2, x_j + \epsilon/2)$, and form the linear combination of characteristic/indicator functions of these half-open intervals

$$g(x) = \sum_{j=1}^{n} F(x_j) \chi_{I_j}(x)\,,$$

which defines a bounded linear functional G on $L^2(\mathbb{R})$ by $G(f) = \int_{\mathbb{R}} f(x) g(x) dx$. Show that if $\lambda < \epsilon/(2 \ln 2)$, then for each j, $G(f_j) > 0$ if $F(x_j) = 1$ and $G(f_j) < 0$ if $F(x_j) = -1$.