Information Theory with Applications

MATH 6397 – Fall 2014

Homework Set 3, due Tuesday, Nov 11, 2014

1. Optimal quantization. Let X be a random variable which is normally distributed with mean zero and variance σ^2. Which map $\phi : \mathbb{R} \to \{a, b\}$ and choice of the two numbers $a < b$ minimizes the mean-square error $\mathbb{E}[(X - \phi(X))^2]$?

2. Entropy maximizers. Prove that among all non-negative random variables with a given mean $\mu > 0$, the exponential distribution has maximal differential entropy.

3. Rate distortion theory for binary random variables.

   (a) Let X be a random variable with values in $\mathbb{A} = \{0, 1\}$ and $\mathbb{P}(X = 0) = p < 1/2$. Assume $0 \le \epsilon \le p$. Show that for any random variable Y with the same alphabet and $\mathbb{P}(X \ne Y) \le \epsilon$, we have

   $$I(X;Y) \ge h(p) - h(\epsilon)$$

   where h is the binary entropy.

   (b) Let X be as before, and assume $0 \le \epsilon \le p < 1/2$. Define a random variable Y such that it has the marginal distribution satisfying

   $$\mathbb{P}(Y = 0) = \frac{1 - p - \epsilon}{1 - 2\epsilon},$$

   and conditioning on the outcome of Y gives $\mathbb{P}(X = 0|Y = 0) = 1 - \epsilon$, $\mathbb{P}(X = 0|Y = 1) = \epsilon$. Prove that

   $$I(X;Y) = h(p) - h(\epsilon)\,.$$

   (c) Given a discrete memoryless source $\{X_j\}_{j=1}^{\infty}$ with alphabet $\mathbb{A} = \{0, 1\}$ and $\mathbb{P}(X_1 = 0) = p < 1/2$, use the preceding parts to show that for the additive Hamming distortion (d_n counts the number of errors for binary input/output sequences of length n),

$$R(D) = \begin{cases} h(p) - h(D), & \text{if } 0 \leq D < p \\ 0, & \text{if } D > p \end{cases}.$$