Information Theory with Applications

MATH 6397 – Fall 2014

November 20, 2014

Homework Set 4, due Tu Dec 9, 2014

1. Erasures and additive noise. Let X be a vector-valued random variable in $\mathbb{R}^k$ with mean zero and covariance matrix $C_X = I$. Assume that X is encoded by an isometry $V : \mathbb{R}^k \to \mathbb{R}^n$, $(VX)_j = \langle X, f_j \rangle$ with an associated (n,k)-frame $\{f_j\}_{j=1}^n$. Now assume that zero-mean noise $N : \Omega \to \mathbb{R}^n$ with covariance $C_N = \sigma^2 I$, $\sigma > 0$, is added to the encoded vector, and that a one-erasure E_l, $l \in \{1, 2, \ldots n\}$ happens each time a vector is sent, with equal probability for each l, and the input vector, the noise and the erasures are independent of each other. Find an expression for the mean-square error

$$\mathrm{MSE}(V) = \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}[\|V^* E_l (VX(\omega) + N(\omega)) - X(\omega)\|^2]$$

   in terms of the frame vectors and determine which choice of (n,k)-frame gives the smallest mean-square error. You may want to introduce $\mathbb{E}_1$ and $\mathbb{E}_2$ as averages with respect to X and N.

   Hints: By the independence of X and N and their vanishing means, if A and B are linear maps from the ranges of X and N to $\mathbb{R}^k$, then $\mathbb{E}[\langle AX(\omega), BN(\omega) \rangle] = 0$. Also, recall that for an (n,k)-frame, $\sum_{j=1}^{n} \|f_j\|^2 = k$.

2. Random coding.

   (a) Write a Matlab function `V=pfgen(n,k)` which generates the analysis operator V for a random (n,k)-frame in the following way: First, choose n random vectors $\{f_j\}_{j=1}^n$ in $\mathbb{R}^k$ with i.i.d. Gaussian entries of variance one, then compute the frame operator S which is given by the matrix with entries

$$S_{k,l} = \sum_{j=1}^{n} \langle e_l, f_j \rangle \langle f_j, e_k \rangle$$

where $k, l \in \{1, 2, \ldots k\}$ and $\{e_l\}_{l=1}^{k}$ is the canonical basis for $\mathbb{R}^k$. With probability one, S will be bounded away from zero, which means we can define vectors $g_j = S^{-1/2} f_j$ for $j \in \{1, 2, \ldots n\}$. These vectors $\{g_j\}_{j=1}^{n}$ then form a Parseval frame for $\mathbb{R}^k$.

(b) Evaluate the performance of random Parseval frames generated according to the method above, consisting of $n \in \{12, 25, 50\}$ vectors in $\mathbb{R}^6$ with a matlab function `pfmse(n)` that does the following: Generate a random $(n, 6)$-frame with `pfgen`, pick 500 zero-mean Gaussian random vectors in $\mathbb{R}^6$ with covariance $C_X = I$ as input, and 500 zero-mean Gaussian noise vectors in $\mathbb{R}^n$ with covariance $C_N = \frac{1}{100} I$ as noise. You may think of these vectors as $X(\omega_r)$ and $N(\omega_r)$, coming from an outcome ω_r indexed by $r \in \{1, 2, \ldots 500\}$. Compute an estimate for the mean square error by averaging over random inputs and noise

$$\widehat{\mathrm{MSE}}(V) = \frac{1}{500n} \sum_{r=1}^{500} \sum_{l=1}^{n} \| V^* E_l (V X(\omega_r) + N(\omega_r)) - X(\omega_r) \|^2$$

for the random frame. Repeat this procedure and average the results over 100 randomly chosen $(n, 6)$-frames to get an estimate $\widehat{\mathrm{MSE}}$ for the expected MSE of a random Parseval frame. Compare the results with the value for the mean-square error of optimal frames from the preceding problem.

Attach your matlab code and the resulting values for the estimated mean-square error.