Functional Analysis, Math 7320
Lecture Notes from August 22, 2016

taken by Bernhard Bodmann

0 Course Information

Text: W. Rudin, Functional Analysis, 2nd edition, McGraw Hill, 1991 (or later).

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Grade: Based on preparation of class notes in LaTeX, rotating note-takers

Background knowledge: Linear algebra, Real analysis, Lebesgue integration

1 Essentials of Topology

1.1 From semimetric to normed spaces, with examples

1.1.1 Definition. Let $\mathcal{X}$ be a set. A map $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+ \equiv [0, \infty[$ is called a *semimetric* on $\mathcal{X}$ if

(1) $d(x, x) = 0$ for all $x \in \mathcal{X}$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$ (symmetry), and

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$ (triangle inequality).

If instead of (1), we have that $d(x, y) = 0$ if and only if $x = y$, then d is called a *metric*.

1.1.2 Example. On $\mathbb{R}$, we define a metric by

$$d(x, y) := \frac{|x - y|}{1 + |x - y|}, \quad x, y \in \mathbb{R}\,.$$

The first two properties follow directly from the definition. To prove the triangle inequality, first show that the function $f : t \mapsto \frac{t}{1+t}$ is subadditive on $\mathbb{R}^+$, so $f(s + t) \leq f(s) + f(t)$ for all $s, t \geq 0$.

Henceforth, we write $\mathbb{K}$ for the field of the real or the complex numbers, $\mathbb{R}$ or $\mathbb{C}$, when either choice is admissible.

1.1.3 Example. On $\mathbb{K}^{\mathbb{N}}$, the space of real or complex sequences, we define a metric by

$$d(x,y) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|},$$

where $x = (x_j)_{j\in\mathbb{N}}$ and $y = (y_j)_{j\in\mathbb{N}}$.

This metric will appear again when we discuss product topologies.

This last example has the natural structure of a vector space. Semimetrics or metrics can arise as a consequence of higher-level structural elements, seminorms or norms.

1.1.4 Definition. Let $\mathcal{V}$ be a vector space over $\mathbb{K}$. A function $p : \mathcal{V} \to \mathbb{R}^+$ is called a *seminorm* if it satisfies

(1) $p(\lambda x) = |\lambda| p(x)$ (homogeneity) for all $\lambda \in \mathbb{K}$, $x \in \mathcal{V}$,

(2) $p(x+y) \leq p(x) + p(y)$ (triangle inequality) for all $x, y \in \mathcal{V}$

If, in addition, $p(x) > 0$ for all $x \in \mathcal{V} \setminus \{0\}$, then p is called a *norm* on $\mathcal{V}$ and $(\mathcal{V}, p)$ a *normed (vector) space*. We often use the notation $\|x\| \equiv p(x)$.

1.1.5 Remark. If p is a seminorm on $\mathcal{V}$, then $d(x,y) := p(x-y)$ defines a semimetric. If p is a norm, then d is a metric.

1.1.6 Example. On $\mathcal{V} = \mathbb{K}^n$, with $n \in \mathbb{N}$, and $p \geq 1$,

$$\|x\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}, \quad x \in \mathcal{V}$$

defines a norm, and so does

$$\|x\|_\infty := \max\{|x_j|\}_{j=1}^n, \quad x \in \mathcal{V}.$$

To prove this, we use *Minkowski's inequality*

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p, \quad x, y \in \mathcal{V}$$

which, in turn, can be shown (exercise) by *Hölder's inequality*

$$|\sum_{j=1}^{n} x_j y_j| \leq \|x\|_p \|y\|_q$$

for $1 \leq p < \infty$, with $q = \begin{cases} \frac{p}{p-1}, & p > 1 \\ \infty, & p = 1 \end{cases}$.

Interpreting $x \in \mathbb{K}^n$ as a map from the index set $\{1, 2, \ldots, n\}$ to $\mathbb{K}$ points to a more general way to generate norms on $\mathbb{K}$-valued function spaces.

1.1.7 Example. Let $\mathcal{X}$ be a set and $B(\mathcal{X}, \mathbb{K})$ the vector space of $\mathbb{K}$-valued bounded functions on $\mathcal{X}$, then

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathcal{X}\}, \quad f \in B(\mathcal{X}, \mathbb{K})$$

defines a norm on this space.

1.1.8 Examples. Other examples of normed spaces of $\mathbb{K}$-valued functions are the sequence spaces

(a) $\ell^p := \{(x_j)_{j\in\mathbb{N}} : \sum_{j=1}^\infty |x_j|^p < \infty\}$ with $\|x\|_p = (\sum_{j=1}^\infty |x_j|^p)^{1/p}$ or

(b) $c_0 := \{(x_j)_{j\in\mathbb{N}} : \lim_{j\to\infty} x_j = 0\}$ with $\|x\|_\infty = \max_{j\in\mathbb{N}} |x_j|$.

Next to the countable index set $\mathbb{N}$, we can also choose an uncountable one.

1.1.9 Example. Let $1 \le p < \infty$, then the space $C([a, b])$ of continuous functions on $[a, b]$ can be given a norm by

$$\|f\|_p := \left(\int_a^b |f(x)|^p dx\right)^{1/p}, \quad f \in C([a, b])\,.$$

To show the axioms, we appeal to *Minkowski's inequality* for (Riemann) integrals

$$\|f + g\|_p \le \|f\|_p + \|g\|_p, \quad f, g \in C([a, b])$$

which can be derived in the same way as for the sequence spaces by *Hölder's inequality*

$$|\int_a^b f(x)g(x)dx| \le \|f\|_p\|g\|_q, \quad f, g \in C([a, b])$$

with $q = \begin{cases} \frac{p}{p-1}, & p > 1 \\ \infty, & p = 1 \end{cases}$ and $\|g\|_\infty = \max\{|g(x)| : x \in [a, b]\}$.

The structure of a semimetric or metric space is reflected in specific sets.

1.1.10 Definition. Let $(\mathcal{X}, d)$ be a semimetric space, then

(a) $B_r(x) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$ is called the *open ball* of rdius $r > 0$ centered at $x \in \mathcal{X}$ and

(b) $U \subset \mathcal{X}$ is called *open* if for each $x \in U$, there is $\epsilon > 0$ such that $B_\epsilon(x) \subset U$.

The notion of open sets can be defined more abstractly without the background of (semi)metric spaces.

1.1.11 Definition. Let $\mathcal{X}$ be a set. The power set $\mathcal{P}(\mathcal{X})$ is the set of all subsets of $\mathcal{X}$. A subset τ of $\mathcal{P}(\mathcal{X})$is called a *topology* on $\mathcal{X}$ if it satisifes

(1) $\emptyset, \mathcal{X} \in \tau$

(2) if $U_1, U_2, \ldots, U_n \in \tau$, then $\cap_{j=1}^n U_j \in \tau$,

(3) if $U_j \in \tau$ for each $j \in J$, then $\cup_{j\in J} U_j \in \tau$.

In this case, $(\mathcal{X}, \tau)$ is called a *topological space* and sets in τ are called *open*.

1.1.12 Examples. For any set $\mathcal{X}$, $\tau = \{\emptyset, \mathcal{X}\}$ or $\tau = \mathcal{P}(\mathcal{X})$ defines a topology. The latter is called the *discrete topology*.

1.1.13 Definition. Let $(\mathcal{X}, \tau)$ be a topological space.

(a) Given $x \in \mathcal{X}$, then $U \subset \mathcal{X}$ is called a *neighborhood of* x if there is $U_0 \subset U$, $U_0 \in \tau$ and $x \in U_0$. We write $\mathcal{U}(x) \equiv \{U : U \text{ is neighborhood of } x\}$.

(b) $F \subset \mathcal{X}$ is *closed* if $X \setminus F$ is open.

With the help of De Morgan's laws, we can deduce properties of closed sets from those of open ones.

1.1.14 Lemma. *Let $(\mathcal{X}, \tau)$ be a topological space, then*

(1) $\emptyset, \mathcal{X}$ are closed,

(2) if $F_1, F_2, \ldots, F_n$ are closed, so is $\cup_{j=1}^n F_j$,

(3) if F_j is closed for each $j \in J$, then so is $\cap_{j \in J} F_j$.

Given a set, we can formulate closed or open sets related to it.

1.1.15 Definition. Let $(\mathcal{X}, \tau)$ be a topological space and $E \subset \mathcal{X}$, then

(a) $\overline{E} = \cap\{F \subset \mathcal{X}, E \subset F, F \text{ closed}\}$ is the *closure* of E,

(b) $E^\circ = \cup\{U \subset \mathcal{X} : U \subset E, U \in \tau\}$ is the *interior* of E, and

(c) $\partial E = \overline{E} \setminus E^\circ$ is its *boundary*.

In most cases, we do not study the most general type of topological spaces.

1.1.16 Definition. A topological space $(\mathcal{X}, \tau)$ is called a *Hausdorff space* if for each $x, y \in \mathcal{X}$, $x \neq y$, we can find neighborhoods of them that are disjoint.

1.1.17 Lemma. *Let $(\mathcal{X}, d)$ be a semimetric space.*

(1) For each $r > 0$, $x \in \mathcal{X}$, the set $B_r(x)$ is open,

(2) For each $r \geq 0$, $x \in \mathcal{X}$, the set $\overline{B}_r(x) \equiv \{y \in \mathcal{X} : d(x,y) \leq r\}$ is closed. It is called the closed ball *of radius r centered at x.*

(3) (X, d) is Hausdorff if and only if d is a metric.

Proof. Exercise. □

1.1.18 Examples. Typical cases of spaces with semimetrics are:

(a) $\mathcal{X} = \mathbb{K}^2$, $d(x,y) = |x_1 - y_1|$,

(b) $\mathcal{X} = C([0,2])$, $d(f,g) = \int_0^1 |f(x) - g(x)| dx$.