Functional Analysis, Math 7320
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1 Essentials of Topology

1.1 Continuity

Next, we recall continuity

1.1.1 Definition. Let X, Y be topological spaces and $x \in X$. A map $f : X \to Y$ is continuous at x, if for each neighborhood U of $f(x)$, $f^{-1}(U)$[1] is a neighborhood of x.

For the purpose of proving results we can weaken the above condition to the following:

1.1.2 Remark. (a) f is continuous at x iff for each neighborhood U of $f(x)$ there exists a neighborhood V of x s.t. $f(V) \subset U$.

(b) If X, Y are metric spaces, then $f : X \to Y$ is continuous at x iff for every $\varepsilon > 0$ there exist a $\delta > 0$ s.t. $f(B_\delta(x)) \subset B_\varepsilon(f(x))$.[2]

1.1.3 Lemma (composition rule)**.** *If $f : X \to Y$ is continuous at x and $g : Y \to Z$ is continuous at $f(x)$, then $g \circ f : X \to Z$ is continuous at x.*

Proof. Follows from definition of continuity and taking inverse image twice. □

1.1.4 Definition. Let X, Y be topological spaces, then $f : X \to Y$ is continuous if for each open set $U \subset Y$, the preimage, $f^{-1}(U)$ is open in X.[3]

1.1.5 Theorem. *Let X, Y be topological spaces and $f : X \to Y$, then TFAE*

(1) f is continuous.

(2) f is continuous at every $x \in X$.

(3) If F is closed in Y, then $f^{-1}(F)$ is closed in X.

(4) For every $M \subset X$, we have that $f(\overline{M}) \subset \overline{f(M)}$

[1] taking inverse is compatible with all set-theoretic operations.
[2] The same result holds for semi-metric spaces.
[3] If $f^{-1}(U) = \emptyset$, then by definition it is still open in X!

Proof. $(1) \iff (2)$ is immediate,

$(1) \implies (3)$,
Let F be closed in Y, then $U = Y \setminus F$ is open in Y, then by (1) $f^{-1}(U)$ is open in X, but $X = f^{-1}(U) \cup f^{-1}(F)$ therefore $f^{-1}(F) = X \setminus f^{-1}(U)$ is closed in X.

$(3) \implies (1)$ The argument is similar.

To prove $(3) \implies (4)$, we first show that $(3) \implies (4')$,

$(4')$ For any $Q \subset Y$, $\overline{f^{-1}(Q)} \subset f^{-1}(\overline{Q})$.

To this end, from $Q \subset \overline{Q}$ we have $f^{-1}(Q) \subset f^{-1}(\overline{Q})$ and since $f^{-1}(\overline{Q})$ is a closed subset of X by (3), $f^{-1}(Q) \subset f^{-1}(\overline{Q}) = \overline{f^{-1}(\overline{Q})}$ and we can take the closure on the left-hand side while the inequality remains intact.

To get from $(4')$ to (4), consider $M \subset X$, and let $Q = f(M)$, then by $(4')$:

$$\overline{f^{-1}(f(M))} \subset f^{-1}(\overline{f(M)}) \;\; (\star)$$

From $M \subset f^{-1}(f(M))$ and $(\star)$ we have that

$$\overline{M} \subset \overline{f^{-1}(f(M))} \subset f^{-1}(\overline{f(M)})$$

Now applying f on both sides gives (4), $f(\overline{M}) \subset \overline{f(M)}$.

Finally, to show $(4) \implies (3)$, Let F be closed in Y, take $E = f^{-1}(F)$.

Let $x \in \overline{E}$. Then we want to show that $x \in E$.

By (4), $f(x) \in \overline{f(E)}$. However, $\overline{f(E)} = \overline{f(f^{-1}(F))} = \overline{F} = F = f(E)$.

So $f(x) \in F$, meaning $x \in f^{-1}(F) = E$.

Thus, $\overline{E} \subset E$, so we get that E is closed. $\square$

1.1.6 Definition. Let $f : X \to Y$ be a map between topological spaces X, Y:

(a) f is a called a "homeomorphism" if it is continuous, invertible, and $f^{-1} : Y \to X$ is also continuous.

(b) f is called "open" if for each open $U \subset X$, $f(U)$ is open in Y.

1.1.7 Remark. In (4), we have if f is a homeomorphism then $f(\overline{M}) = \overline{f(M)}$. However, the converse is not true as the following example shows.

1.1.8 Example. Consider the projection mapping $f:\mathbb{R}^2 \to \mathbb{R}$ where $f(x,y)=y$. Then f is not a bijection. However, for any $M \subset \mathbb{R}^2$ we have $f(\overline{M}) = \overline{f(M)}$.

On the other hand, if f is continuous and one-to-one, but f^{-1} is not continuous, then we do not necessarily get equality in (4):

1.1.9 Example. Let $I : (\mathbb{R}, D) \to (\mathbb{R}, d)$ be the identity map, where D stands for the discrete metric, and d stands for the usual metric, then wth respect to D, $B_1(0) = \{0\}$ and thus $I(\overline{B_1(0)}) = I(B_1(0)) = \{0\} \subsetneq \overline{I(B_1(0))} = \overline{B_1(0)} = \overline{B}_1(0)$.

1.1.10 Lemma. *Let $f : X \to Y$, $g : Y \to Z$ be continuous, then so is $g \circ f : X \to Z$.*

Proof. Follows directly from the definition and taking inverse images twice. $\square$

1.1.11 Lemma. *Let $f : X \to Y$ be continuous and invertible, then f is open iff it is a homeomorphism.*

Proof. Follows straightly from the definition. $\square$

1.1.12 Definition. Let X, Y be normed vector spaces and $A : X \to Y$ be linear. We define the operator norm of A by

$$\|A\| = \text{sup}\,\{\|Ax\| : \|x\| \leq 1\} \in [0, \infty]$$

The map A is called "bounded" if $\|A\| < \infty$. We write $\mathcal{B}(X, Y)$ for the set of all such linear maps.

1.1.13 Lemma. *(a)* $(\mathcal{B}(X, Y), \|.\|)$ *is a normed vector space.*

(b) If $A \in \mathcal{B}(X, Y)$, $x \in X$, *then*
$\|Ax\| \leq \|A\|.\|x\|$, *and*
$\|A\| = \inf\{c > 0 : \forall x \in X, \|Ax\| \leq c\|x\|\}$

Proof. (a) Follows straightly from the definition.

(b) For the first inequality we have

$\|A(\frac{x}{\|x\|})\| \leq \sup\{\|Ay\| : \|y\| \leq 1\} = \|A\|$ therefore by linearity

$\|Ax\| \leq \|A\|.\|x\|$

To see the second inequality, let $\{x_j\}_{j\in\mathbb{N}}$ be s.t. $\|x_j\| \leq 1$ for each $j \in \mathbb{N}$, and $\|Ax_j\| \to \|A\|$[4]. Hence, if there is positive C[5] for which $\|Ax\| \leq C\|x\|$ for each $x \in X$, then in particular $\|Ax_j\| \leq C\|x_j\|$ for each $j \in \mathbb{N}$, and therefore

[4]Existance of such a sequence follows from the definition of supremum.
[5]Since A is bounded such a positive C exists.

$\|A\| = \lim_{j\to\infty} \|Ax_j\| \leq \limsup_{j\to\infty} C\|x_j\| \leq C$

Consequently, by taking infimum over all such positive C we have,
$\|A\| \leq \inf\{C > 0 : \forall x \in X, \|Ax\| \leq C\|x\|\}$

Next, by $\|Ax\| \leq \|A\|.\|x\|$ for all $x \in X$, taking $C = \|A\|$, we have:
$\inf\{C > 0 : \forall x \in X, \|Ax\| \leq C\|x\|\} \leq \|A\|$,
So equality hold.

□

The following example show that in an arbitrary metric space we do not necessarily have, "The closure of unit ball" = "The closed unit ball".

1.1.14 Example. Take $X = \mathbb{R}$ with the discrete metric, i.e. $d(x, y) = \delta(x, y)$, then:

$B_1(0) = \{0\}$, therefore $\overline{B_1(0)} = \{0\}$. But $\overline{B}_1(0) = \mathbb{R}$. Therefore $\overline{B_1(0)} \subsetneq \overline{B}_1(0)$.

However, in normed spaces this pathological type of example does not occur.

1.1.15 Proposition. *In a normed vector space X, we always have $\overline{B_1(0)} = \overline{B}_1(0)$.*

Proof. (1) $\overline{B_1(0)} \subset \overline{B}_1(0)$.

Since $B_1(0) \subset \overline{B}_1(0)$, and $\overline{B}_1(0)$ is closed, by the minimality of the closure, $\overline{B_1(0)} \subset \overline{B}_1(0)$.

(2) $\overline{B}_1(0) \subset \overline{B_1(0)}$.

We show if $x \in \overline{B}_1(0)$ then $x \in \overline{B_1(0)}$. If $\|x\| < 1$ then we have nothing to show. Next, let $x \in \overline{B}_1(0) \setminus B_1(0)$, so $\|x\|_X = 1$. Then the sequence $x_n := (1 - 1/n)x$ is in $B_1(0)$, and converges to x in the norm topology of X. Therefore $x \in \overline{B_1(0)}$, which completes the proof.

□

1.1.16 Theorem. *Given a linear map $A : X \to Y$ between normed vector spaces, TFAE:*

(1) A is continuous.

(2) A is bounded.

(3) A is continuous at 0.

Proof. To distinguish balls in different spaces, we write $B_r^X(p)$ for a ball in X, and $B_r^Y(p)$ for a ball in Y.

(1) $\implies$ (2) Given (1), $B_1^Y(0)$, then by continuity $A^{-1}(B_1^Y(0)) = V$ is open in X, and $0 \in V$, so there is $r > 0$ s.t. $A(B_r^X(0)) \subset B_1^Y(0)$. By scaling/ linearity, $A(B_1^X(0)) \subset B_{\frac{1}{r}}^Y(0)$.

Using the Characterization of continuity with closures, $A(\overline{B_1^X(0)}) \subset \overline{A(B_1^X)(0)}$. However, by the previous remark we also have:

$$\overline{B_1^X(0)} = \overline{B}_1^X(0), \text{ and } \overline{B_{\frac{1}{r}}^Y(0)} = \overline{B}_{\frac{1}{r}}^Y(0)$$

Therefore $A(\overline{B}_1^X(0)) \subset \overline{B}_{\frac{1}{r}}^Y(0)$ and we conclude that $\|A\| \leq \frac{1}{r}$, so A is bounded.

(2) $\implies$ (3) Next, assume A is bounded, therefore $\|A\| \leq \frac{1}{r}$ for some $r > 0$. Let $\varepsilon > 0$ be given. Consider $B_\varepsilon^Y(0)$, then take $\delta = \frac{r\varepsilon}{2}$. Then, by scaling we get

$$A(B_\delta^X(0)) \subset A(\overline{B}_\delta^X(0)) = \delta A(\overline{B}_1^X(0)) \subset \text{ (by assumption) } \delta \overline{B}_{1/r}^Y(0) = \overline{B}_{\varepsilon/2}^Y(0) \subset B_\varepsilon^Y(0).$$

So A is continuous at 0.

(3) $\implies$ (1) Finally, let A be continuous at 0. We want to show it is continuous at each $x \in X$, and hence continuous.

Given $x \in X$, and $B_\varepsilon^Y(Ax)$, take $\delta > 0$ s.t. $A(B_\delta^X(0)) \subset B_\varepsilon^Y(0)$. By linearity, we have:

$$A(B_\delta^X(x)) = A(x + B_\delta^X(0)) = Ax + A(B_\delta^X(0)) \subset Ax + B_\varepsilon^Y(0) = B_\varepsilon^Y(Ax)$$

Hence A is continuous at x.

$\square$