# Functional Analysis, Math 7320
# Lecture Notes from September 1, 2016

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*The following items are the continuation of a list of examples of the previous set of notes:*

(e') If $X$ is a metric space, then $(C_b(X,\mathbb{K}), \|\cdot\|_\infty)$ is a Banach space, where $C_b(X,\mathbb{K})$ is the set of continuous, bounded, $\mathbb{K}$-valued functions on $X$ and $\|\cdot\|_\infty$ is defined by $f \mapsto \sup_{x\in X}|f(x)|$.

To see that $\|\cdot\|_\infty$ is a norm, observe that

$$\|f\|_\infty = \sup_{x\in X}|f(x)| = 0 \iff f(x) = 0 \text{ for all } x \in X.$$

Moreover, if $\lambda \in \mathbb{R}$, then $\|\lambda f\|_\infty = \sup_{x\in X}|\lambda f(x)| = |\lambda|\sup_{x\in X}|f(x)| = |\lambda|\|f\|_\infty$.

Finally, if $f, g \in C_b(X,\mathbb{R})$, then

$$\|f+g\|_\infty = \sup_{x\in X}|f(x)+g(x)| \leq \sup_{x\in X}|f(x)| + \sup_{x\in X}|g(x)| = \|f\|_\infty + \|g\|_\infty.$$

To see that $(C_b(X,\mathbb{R}), \|\cdot\|_\infty)$ is complete, let $(f_n)_{n\in\mathbb{N}}$ be Cauchy in $C_b(X,\mathbb{R})$ and let $\varepsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon$ whenever $m, n \geq n_0$, which implies that for all $x \in X$, $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty < \varepsilon$ whenever $m, n \geq n_0$, which in turn implies that $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ and thus converges to an element of $\mathbb{R}$. Define $f : X \to \mathbb{R}$ by $x \mapsto \lim_{n\to\infty} f_n(x)$.

To see that $(f_n)_{n\in\mathbb{N}}$ converges to $f$ and that $f$ is continuous, let $\varepsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $\|f_m - f_n\|_\infty < \varepsilon/2$ whenever $m, n \geq n_0$, which implies that for all $x \in X$, $|f_m(x) - f_n(x)| < \varepsilon/2$ whenever $m, n \geq n_0$, which in turn implies that for all $x \in X$,

$$\lim_{m\to\infty}|f_m(x) - f_n(x)| = |f(x) - f_n(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

whenever $n \geq n_0$. Therefore, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f$, which implies that $f$ is continuous.

To see that $f$ is bounded, let $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that $\|f - f_{n_0}\| < \varepsilon$, and note that $f_{n_0}$ is bounded, which implies that there is an $M \geq 0$ such that $\|f_{n_0}\|_\infty \leq M$. Then $\|f\|_\infty \leq \|f - f_{n_0}\|_\infty + \|f_{n_0}\|_\infty < \varepsilon + M$. Therefore, $f$ is bounded.

(f) If $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a Banach space, then $(\mathcal{B}(X,Y), \|\cdot\|_{\mathsf{op}})$ is a Banach space, where $\|\cdot\|_{\mathsf{op}}$ is the operator norm defined by

$$S \mapsto \sup_{x \neq 0} \frac{\|Sx\|_Y}{\|x\|_X}.$$

To see that $\mathcal{B}(X,Y)$ is a normed space, let $S, T \in \mathcal{B}(X,Y)$. Then

$$\|S+T\|_{\mathsf{op}} = \sup_{x \neq 0} \frac{\|(S+T)x\|_Y}{\|x\|_X} \leq \sup_{x \neq 0} \frac{\|Sx\|_Y}{\|x\|_X} + \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \|S\|_{\mathsf{op}} + \|T\|_{\mathsf{op}}.$$

Let $\lambda \in \mathbb{K}$. Then

$$\|\lambda S\|_{\mathsf{op}} = \sup_{x \neq 0} \frac{\|\lambda Sx\|_Y}{\|x\|_X} = |\lambda| \sup_{x \neq 0} \frac{\|Sx\|_Y}{\|x\|_X} = |\lambda| \|S\|_{\mathsf{op}}.$$

Observe that

$$0 = \|T\|_{\mathsf{op}} = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \iff \|Tx\|_Y = 0 \text{ for all } 0 \neq x \in X \iff T = 0.$$

To see that $\mathcal{B}(X,Y)$ is complete, let $(T_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(X,Y)$, let $\varepsilon > 0$, and let $x \in X$. Then there is an $n_0 \in \mathbb{N}$ such that $\|T_m - T_n\|_{\mathsf{op}} < \varepsilon/\|x\|_X$ whenever $m, n \geq n_0$, which implies that $\|T_m x - T_n x\|_Y \leq \|T_m - T_n\|_{\mathsf{op}} \|x\|_X < \varepsilon$, which in turn implies that $(T_n x)_{n \in \mathbb{N}}$ is a Cauchy sequence in $Y$ and thus converges to an element in $Y$ since $Y$ is complete. Therefore, define $T : X \to Y$ by $x \mapsto \lim_{n \to \infty} T_n x$.

To see that $T$ is linear, let $x, y \in X$ and let $\lambda \in \mathbb{K}$. Then

$$T(\lambda x + y) = \lim_{n \to \infty} T_n(\lambda x + y) = \lambda \lim_{n \to \infty} T_n x + \lim_{n \to \infty} T_n y = \lambda Tx + Ty.$$

To see that $T$ is bounded, let $\varepsilon = 1$ and let $x \in X$ such that $\|x\|_X \leq 1$. Then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\|Tx\|_Y \leq \|T_n x\|_Y + \|Tx - T_n x\|_Y < \|T_n\|_{\mathsf{op}} + 1.$$

To see that $\lim_{n \to \infty} T_n = T$, let $\varepsilon > 0$ and let $x \in X$ such that $\|x\|_X \leq 1$. Then there is an $n_0 \in \mathbb{N}$ such that $\|Tx - T_n x\|_Y < \varepsilon/2$ whenever $n \geq n_0$. Since $(T_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}(X,Y)$, there is an $N_0 \in \mathbb{N}$ such that $\|T_m - T_n\|_{\mathsf{op}} < \varepsilon/2$ whenever $m, n \geq N_0$. Therefore,

$$\|Tx - T_n x\|_Y \leq \|Tx - T_m x\|_Y + \|T_m x - T_n x\|_Y < \varepsilon$$

whenever $m, n \geq \max\{n_0, N_0\}$.

## 1.3 Completion

**1.3.1 Definition.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f : X \to Y$.

(1) If $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$, then $f$ is called an **isometry**.

(2) If $f(X) = Y$ and $f$ is an isometry, then $f$ is called an **isometric isomorphism**, and we write $X \cong Y$.

(3) If $f$ is an isometry, $\overline{f(X)} = Y$, and $Y$ is complete, then $f$ is called a **completion**.

**1.3.2 Theorem.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces.*

*(1) $X$ has a completion.*

*(2) If $f : X \to Y$ is uniformly continuous, then there is a unique uniformly continuous map $\hat{f} : \hat{X} \to \hat{Y}$ such that $\hat{f}|_X = f$, where $\eta : X \to (\hat{X}, d_{\hat{X}})$ and $\varphi : Y \to (\hat{Y}, d_{\hat{Y}})$ are completions.*

*(3) If $X$ has completions $\hat{X}$ and $\eta : X \to Y$, then $\hat{X}$ and $Y$ are isometrically isomorphic.*

*Proof.* (1) Fix $x_0 \in X$ and define $\eta : X \to C_b(X, \mathbb{R})$ by $x \mapsto f_x$, where $f_x : X \to \mathbb{R}$ is defined by $y \mapsto d_X(x, y) - d_X(y, x_0)$. Recall that $C_b(X, \mathbb{R})$ is a metric space with the metric induced by $\| \cdot \|_\infty$. Since $f_x(y) = d_X(x, y) - d_X(y, x_0) \leq d_X(x, x_0)$, $f_x$ is bounded. Since $d_X$ is continuous, $f_x$ is continuous. Therefore, $f_x \in C_b(X, \mathbb{R})$. Let $x_1, x_2, y \in X$. Then $f_{x_1}(y) - f_{x_2}(y) = d_X(x_1, y) - d_X(y, x_2)$, which implies that $|f_{x_1}(y) - f_{x_2}(y)| \leq d_X(x_1, x_2)$. Observe that equality is attained if $y = x_2$. Therefore, $\|f_{x_1} - f_{x_2}\|_\infty = d_X(x_1, x_2)$, which implies that $\eta$ is an isometry. Let $\hat{X} = \overline{\eta(X)}$. Since $(C_b(X, \mathbb{R}), \| \cdot \|_\infty)$ is a Banach space and $\hat{X}$ is closed, $\hat{X}$ is complete, which implies that $\hat{X}$ is a completion of $X$.

(2) Identify $X$ with $\eta(X)$, identify $Y$ with $\varphi(Y)$, let $\hat{x} \in \hat{X}$, and let $\varepsilon > 0$. Since $X$ is dense in $\hat{X}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ that converges to $\hat{x}$. Since $f$ is uniformly continuous, there is a $\delta > 0$ such that for all $x, x' \in X$ with $d_{\hat{X}}(x, x') < \delta$, $d_{\hat{Y}}(f(x), f(x')) < \varepsilon$. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, there is an $n_0 \in \mathbb{N}$ such that $d_{\hat{X}}(x_m, x_n) < \delta$ whenever $m, n \geq n_0$, which implies that $d_{\hat{Y}}(f(x_m), f(x_n)) < \varepsilon$ whenever $m, n \geq n_0$, which in turn implies that $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy and thus converges to an element of $\hat{Y}$. Extend $f$ to $\hat{f}$ by defining $\hat{f}(\hat{x}) = \lim_{n\to\infty} f(x_n)$.

To see that $\hat{f}$ is well defined, let $(x'_n)_{n\in\mathbb{N}}$ be a sequence in $X$ that converges to $\hat{x}$. Then, by the previous argument, $(\hat{f}(x'_n))_{n\in\mathbb{N}}$ is Cauchy, which implies that $(\hat{f}(x_1), \hat{f}(x'_1), \hat{f}(x_2), \hat{f}(x'_2), \dots)$ is Cauchy and converges to $\hat{x}$ since its subsequence $(\hat{f}(x_n))_{n\in\mathbb{N}}$ converges to $\hat{x}$, which in turn implies that $(\hat{f}(x'_n))_{n\in\mathbb{N}}$ converges to $\hat{x}$.

To see that $\hat{f}|_X = f$, let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be such that $x_n = x$ for all $n \in \mathbb{N}$. Then $\hat{f}(x) = \lim_{n\to\infty} f(x_n) = f(x)$.

To see that $\hat{f}$ is uniformly continuous, let $\varepsilon > 0$. Since $\hat{f}|_X$ is uniformly continuous, there is a $\delta > 0$ such that for all $x, y \in X$ with $d_{\hat{X}}(x, y) < \delta$, $d_{\hat{Y}}(\hat{f}(x), \hat{f}(y)) < \varepsilon/3$. Let $\hat{x}, \hat{y} \in \hat{X}$ with $d_{\hat{X}}(\hat{x}, \hat{y}) < \delta/3$. Then there are sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $X$ converging to $\hat{x}$

and $\hat{y}$, respectively, which implies that there is an $n_0 \in \mathbb{N}$ such that $d_{\hat{X}}(x_n, \hat{x}) < \delta/3$ and $d_{\hat{X}}(y_n, \hat{y}) < \delta/3$ whenever $n \geq n_0$. Since $d_{\hat{X}}(x_n, y_n) \leq d_{\hat{X}}(x_n, \hat{x}) + d_{\hat{X}}(\hat{x}, \hat{y}) + d_{\hat{X}}(\hat{y}, y_n) < \delta$ whenever $n \geq n_0$, $d_{\hat{Y}}(\hat{f}(x_n), \hat{f}(y_n)) < \varepsilon/3$ whenever $n \geq n_0$. Since $(\hat{f}(x_n))_{n\in\mathbb{N}}$ and $(\hat{f}(y_n))_{n\in\mathbb{N}}$ converge to $\hat{f}(\hat{x})$ and $\hat{f}(\hat{y})$, respectively, there is an $N_0 \in \mathbb{N}$ such that $d_{\hat{Y}}(\hat{f}(x_n), \hat{f}(\hat{x})) < \varepsilon/3$ and $d_{\hat{Y}}(\hat{f}(y_n), \hat{f}(\hat{y})) < \varepsilon/3$ whenever $n \geq N_0$. Letting $N = \max\{n_0, N_0\}$ yields that $d_{\hat{Y}}(\hat{f}(\hat{x}), \hat{f}(\hat{y})) \leq d_{\hat{Y}}(\hat{f}(\hat{x}), \hat{f}(x_n)) + d_{\hat{Y}}(\hat{f}(x_n), \hat{f}(y_n)) + d_{\hat{Y}}(\hat{f}(y_n), \hat{f}(\hat{y})) < \varepsilon$.

To see that $\hat{f}$ is unique, let $h : \hat{X} \to \hat{Y}$ be a uniformly continuous extension of $f$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X$ converging to $\hat{x} \in \hat{X}$. Since $h$ is continuous, $\lim_{n\to\infty} h(x_n) = h(\lim_{n\to\infty} x_n) = h(\hat{x})$. Since $h(x_n) = f(x_n)$ for all $n \in \mathbb{N}$, $h(\hat{x}) = \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} f(x_n) = \hat{f}(\hat{x})$.

$(3)$ Identify $X$ with $\eta(X)$. To see that $\eta$ is uniformly continuous, let $\varepsilon = \delta > 0$ and observe that if $x, y \in X$ are such that $d_X(x, y) < \delta$, then $d_Y(\eta(x), \eta(y)) = d_X(x, y) < \delta = \varepsilon$ since $\eta$ is an isometry. Moreover, observe that the completion of a complete metric space is its identity map. Therefore, $\eta$ extends to $\hat{\eta} : \hat{X} \to Y$.

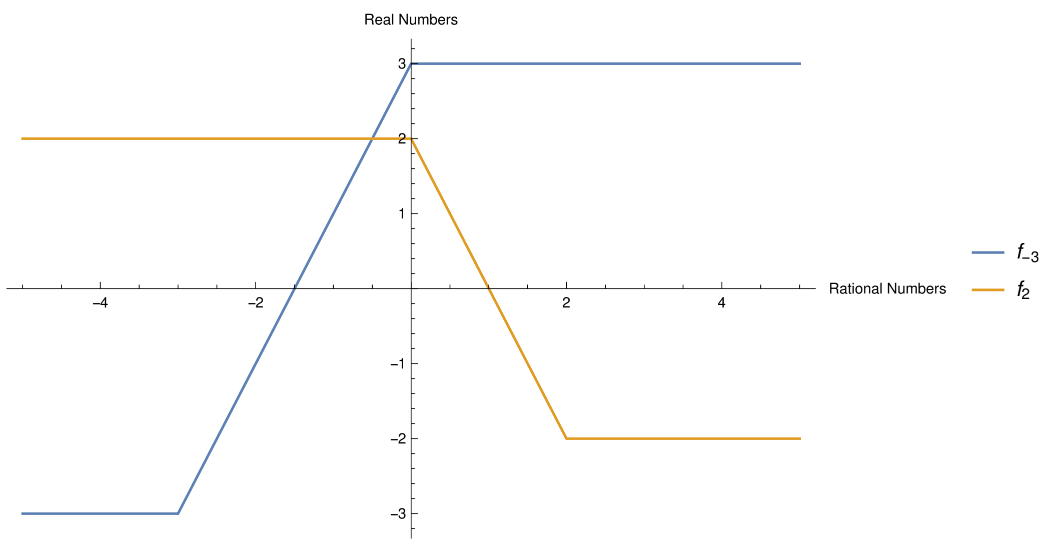
To see that $\hat{\eta}$ is an isometry, let $\hat{x}, \hat{y} \in \hat{X}$. Then there are sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $X$ that converge to $\hat{x}$ and $\hat{y}$, respectively. Therefore,

$$d_{\hat{X}}(x_n, y_n) = d_X(x_n, y_n) = d_Y(\eta(x_n), \eta(y_n)) = d_{\hat{Y}}(\eta(x_n), \eta(y_n)),$$

and taking the limit as $n$ approaches infinity yields that $d_{\hat{X}}(\hat{x}, \hat{y}) = d_{\hat{Y}}(\eta(\hat{x}), \eta(\hat{y}))$.

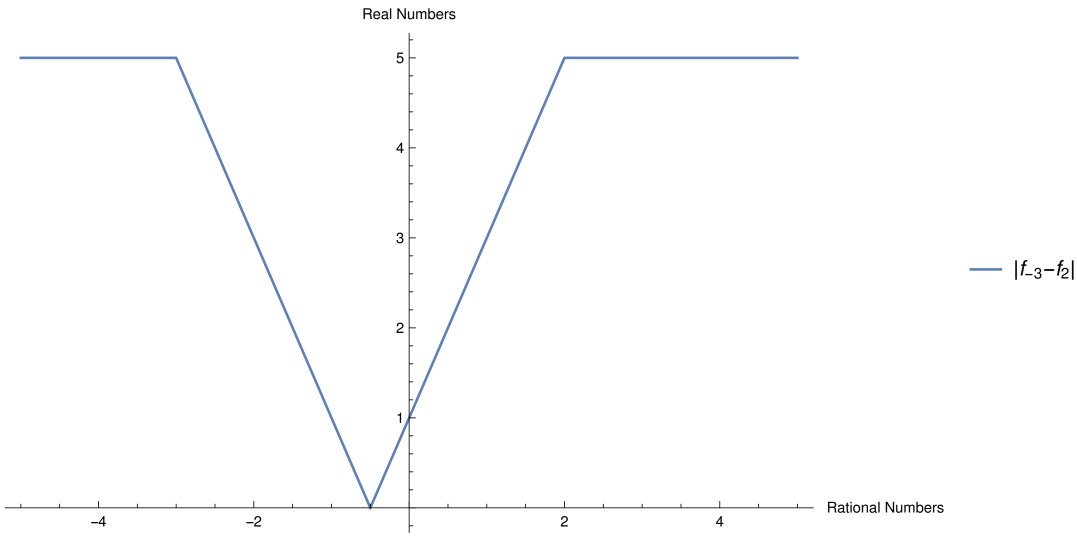
As above, $\eta^{-1} : \eta(X) \to X$ extends to an isometry $\widehat{\eta^{-1}} : Y \to \hat{X}$. Therefore, $\widehat{\eta^{-1}} \circ \hat{\eta} : \hat{X} \to \hat{X}$ is an extension of the identity map on $X$, which implies that $\widehat{\eta^{-1}} \circ \hat{\eta} : \hat{X} \to \hat{X}$ is the identity map on $\hat{X}$. Similarly, $\hat{\eta} \circ \widehat{\eta^{-1}} : Y \to Y$ is the identity map on $Y$. Therefore, $\hat{\eta}$ is an isometric isomorphism. □

To visualize the functions $f_x$ constructed in the proof of $(1)$ above, let $X = \mathbb{Q}$ and let $x_0 = 0$. Then the graphs of $f_{-3}$ and $f_2$ are as follows:




Plots of $f_{-3}$ and $f_2$ with $x_0=0$.

Moreover, the graph of $|f_{-3} - f_2|$ is as follows:




Plot of $|f_{-3}-f_2|$ with $x_0=0$.

Therefore, $\sup_{x\in\mathbb{Q}} |f_{-3}(x) - f_2(x)| = 5$, which shows us that $|-3-2| = \|f_{-3} - f_2\|_\infty$.