Functional Analysis, Math 7320
Lecture Notes from Septempber 6, 2016

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Recall the theorem from the previous lecture:

1.3.2 Theorem. *Let X be a metric space. Then*

(i) X has a completion $\eta : X \to \hat{X}$.

(ii) If $f : X \to Y$ is uniformly continuous and Y has a completion $\hat{Y}$, then there exists a unique (uniformly) continuous map $\hat{f} : \hat{X} \to \hat{Y}$ such that $\hat{f}|_X = f$.

(iii) If $\eta : X \to \hat{X}$ and $f : X \to Y$ are completions, then $\hat{f} : \hat{X} \to Y$ is isometric isomorphism.

1.3.3 Remarks. (1) Since a metric space X is dense in its completion $\hat{X}$, we can identify an element of $x \in \hat{X}$ to be a sequence (x_n) in X such that $\lim_{n\to\infty} x_n = x$.

(2) Let $x, y \in \hat{X}$ and suppose that (x_n) and (y_n) are sequences in X converging to x and y respectively. If d is a metric on X and $\hat{d}$ is the extension of the metric d in $\hat{X}$. We have $\hat{d}(x_n, y_n) = d(x_n, y_n)$ for $n \in \mathbb{N}$. Thus,

$$\begin{aligned}|d(x_n, y_n) - \hat{d}(x, y)| &= |\hat{d}(x_n, y_n) - \hat{d}(x_n, y) + \hat{d}(x_n, y) - \hat{d}(x, y)| \\ &\leq |\hat{d}(x_n, y_n) - \hat{d}(x_n, y)| + |\hat{d}(x_n, x) - \hat{d}(x, y)| \\ &\text{(By the property of a metric: } d(a,b) - d(a,c) \leq d(b,c)\text{)} \\ &\leq |\hat{d}(y_n, y)| + |\hat{d}(x_n, x)|\end{aligned}$$

Since $x_n \to x$ and $y_n \to y$ in $\hat{X}$, $|d(x_n, y_n) - \hat{d}(x, y)| \to 0$ as $n \to \infty$, i.e., $\hat{d}(x, y) = \lim_{n\to\infty} d(x_n, y_n)$.

(3) The metric in a normed space $(X, \|\cdot\|)$ is $d(a, b) = \|a - b\|$ for $a, b \in X$. Following from the above,

$$\hat{d}(x, y) = \lim_{n\to\infty} \|x_n - y_n\|$$

is the metric on $\hat{X}$.

(4) Let $x \in \hat{X}$ and (x_n) be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then, $\lim_{n\to\infty} \lambda x_n = \lambda x$.

Proof. As $n \to \infty$,

$$\hat{d}(\lambda x_n, \lambda x) = \lim_{n\to\infty} \|\lambda x_n - \lambda x\| = \lim_{n\to\infty} |\lambda| \|x_n - x\| = |\lambda| \lim_{n\to\infty} \|x_n - x\| \to 0.$$

□

We know that a normed space is a metric space. Thus, it has a completion. Interesting questions are that "whether the completion of a normed space is a normed space" and "whether the extension of a continuous linear is a linear map." These two questions will be answered by the next two corollaries.

1.3.4 Corollary. *If $(X, \|\cdot\|)$ is a normed space, then its completion $\hat{X}$ is a Banach space in which X is dense.*

Proof. By the previous theorem, X has a completion $\hat{X}$ and X is dense in $\hat{X}$. We need to show that $\hat{X}$ is also a normed space. (i) First, we show that $\hat{X}$ is a vector space. Let $x, y \in \hat{X}$ and $\lambda \in \mathbb{K}$. There are sequences (x_n) and (y_n) in X converging to x and y respectively. Define

$$x + y = \lim_{n\to\infty} (x_n + y_n),$$

and

$$\lambda x = \lim_{n\to\infty} \lambda x_n.$$

The definition is independent with the choice of sequences. Because (x_n) and (y_n) are Cauchy sequences, its addition is a Cauchy sequence. Also, by multiplying with a scalar with a sequence, it is also a Cauchy sequence. Since $\hat{X}$ is complete, limits of those sequence exist. Thus, the above definition is well-defined. Additive identity and scalar multiplication identity of $\hat{X}$ are obtained directly from the identities of X. The fact that $\hat{X}$ is a vector space follow from the properties of limit. For example, we show that $x + y = y + x$.

$$x + y = \lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} (y_n + x_n) = y + x.$$

The other axioms can be proved similarly. (ii) Since $\|\cdot\| : X \to \mathbb{R}$ is uniformly continuous, by the extension of (uniformly) continuous of $\|\cdot\|$, we have $\|\hat{\cdot}\| : \hat{X} \to \mathbb{R}$ is also uniformly continuous. By the continuity of $\|\hat{\cdot}\|$, we can pass the limit inside or outside $\|\hat{\cdot}\|$. Thus, for any $x \in \hat{X}$ and (x_n) a sequence in X converging to x,

$$\|\hat{x}\| = \|\lim_{n\to\infty}\hat{} \, x_n\| = \lim_{n\to\infty} \|\hat{x_n}\| = \lim_{n\to\infty} \|x_n\|.$$

The fact that $\|\hat{\cdot}\|$ is a norm following from the limit properties and norm axioms of $\|\cdot\|$. For instant, we show that $\|x \hat{+} y\| \leq \|\hat{x}\| + \|\hat{y}\|$. Let $x, y \in \hat{X}$. Then, by the density of X in $\hat{X}$, there are sequences (x_n) and (y_n) in X converging to x and y respectively. Then,

$$\|x \hat{+} y\| = \lim_{n\to\infty} \|x_n + y_n\| \leq \lim_{n\to\infty} (\|x_n\| + \|y_n\|) = \lim_{n\to\infty} \|x_n\| + \lim_{n\to\infty} \|y_n\| = \|\hat{x}\| + \|\hat{y}\|.$$

Define $\tilde{d}(x, y) = \|x \hat{-} y\|$, by the remark (1.3.3) number (3), both $\tilde{d}$ and $\hat{d}$ coincide. $\square$

1.3.5 Corollary. *If $A : X \to Y$ is a continuous linear map from a normed space X to a Banach space Y, then there is a unique linear extension $\hat{A} : \hat{X} \to Y$.*

Proof. Since a linear continuous map is uniformly continuous, there is a unique continuous map $\hat{A} : \hat{X} \to Y$. Since $\hat{A}$ is continuous, we can pass through the limit inside or outside the map $\hat{A}$. Therefore, for any $x \in \hat{X}$ and (x_n) a sequence in X converging to x,

$$\hat{A}(x) = \hat{A}(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \hat{A}(x_n) = \lim_{n\to\infty} A(x_n).$$

We still need to show that the map $\hat{A}$ is a linear map. Let $x, y \in \hat{X}$ and $\lambda \in \mathbb{K}$. Then, by the density of X in $\hat{X}$, there is a sequence (x_n) and (y_n) in X converge to x and y respectively. Thus, by limit properties and linearity of A, we have

$$\begin{aligned}\hat{A}(\lambda x + y) &= \lim_{n\to\infty} (A(\lambda x_n + y_n)) = \lim_{n\to\infty} (\lambda A(x_n) + A(y_n)) = \lambda \lim_{n\to\infty} A(x_n) + \lim_{n\to\infty} A(y_n) \\ &= \lambda \hat{A}(x) + \hat{A}(y).\end{aligned}$$

□

1.3.6 Remark. Norms of the linear map A and its extension $\hat{A}$ from the previous corollary are equal. To avoid confusion, we denote

- $\|\cdot\|_X$ is the norm in X, and $\|\cdot\|_{\hat{X}}$ is its extended complete norm in $\hat{X}$, and $\|\cdot\|_Y$ is the norm in Y.
- A is the linear map in X, and $\hat{A}$ is the extended linear map in $\hat{X}$.

We want to show that $\|A\|_X = \|\hat{A}\|_{\hat{X}}$.

Proof. Let $x \in \hat{X}$ and (x_n) be a sequence in X converging to x. Then, for all $n \in \mathbb{N}$,

$$\|A(x_n)\|_Y \leq \|A\|_X \|x_n\|_X.$$

By continuity of $\|\cdot\|$, by taking limit $n \to \infty$ in the previous inequality,

$$\lim_{n\to\infty} \|A(x_n)\|_Y = \|\lim_{n\to\infty} A(x_n)\| = \|\hat{A}(x)\|_Y \leq \lim_{n\to\infty} \|A\|_X \|x_n\|_X = \|A\|_X \|x\|_{\hat{X}}.$$

Thus, $\|\hat{A}\|_{\hat{X}} \leq \|A\|_X$. Conversely, for any $x \in X$, we have $\|x\|_X = \|x\|_{\hat{X}}$ and $\|A(x)\|_X = \|\hat{A}(x)\|_{\hat{X}}$. Since supreme in a smaller set is less than a bigger one, and $X \subset \hat{X}$,

$$\|A\|_X = \sup_{0\neq x\in X} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{0\neq x\in X} \frac{\|\hat{A}(x)\|_Y}{\|x\|_{\hat{X}}} \leq \sup_{0\neq x\in \hat{X}} \frac{\|\hat{A}x\|_Y}{\|x\|_{\hat{X}}} = \|\hat{A}\|_{\hat{X}}.$$

In conclusion, $\|A\| = \|\hat{A}\|_{\hat{X}}$. □

In many cases, a function on a completion of a normed space is abstract and difficult to study. Thus, it is practical to determine it by a function of the normed space that extends to that function instead. Most behavior of a function on Banach space can be determined by a function on a normed space that extends to that Banach space.

1.3.7 Examples. (a) Let $X = C_c(\mathbb{R}^n)$, the space of continuous function with compact support on $\mathbb{R}^n$, i.e., supp$\{f\} = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$ is bounded. Then,

$$\|f\|_p := (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$$

is a norm. Then, the completion of $C_c(\mathbb{R}^n)$ is called $L^p(\mathbb{R}^n)$.

(b) Let $f \in C([a,b])$ and $K : [a,b] \times [a,b] \to \mathbb{K}$ be continuous, then $T_k : f \mapsto T_K f$ defined by

$$T_K f(x) = \int_a^b K(x,y) f(y) dy$$

is continuous with respect to the norm $\| \cdot \|_p$. We can extend it to $\hat{T_K} : L^p([a,b]) \to L^p([a,b])$.

Proof. First, we show that $Tf(x)$ is continuous.

$$|Tf(x) - Tf(x_0)| = |\int_a^b (K(x,y) - K(x_0,y)) f(y) dy|.$$

Since K is a continuous on compact set, it is also a uniformly continuous. Thus, for $\varepsilon > 0$, and $|x - x_0|$ is sufficiently small, $|K(x,y) - K(x_0,y)| < \varepsilon$ for every $y \in [a,b]$. Since f is continuous on a compact set $[a,b]$, there is $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Thus, $|\int_a^b f(y) dy| \leq \int_a^b M dx = M|b-a|$ for some $M \geq 0$. Thus,

$$|Tf(x) - Tf(x_0)| \leq |\varepsilon \int_a^b f(y) dy| \leq M|b-a|\varepsilon.$$

Thus, $T : C([a,b]) \to C([a,b])$. By the linearity of integration, T is linear map. To show that T is continuous with respect to $\| \cdot \|_p$, we show that it is a bounded operator. For $f \in C([a,b])$ such that $\|f\|_p = 1$,

$$(\|Tf\|_p)^p = \int_a^b |Tf(x)|^p dx = \int_a^b |\int_a^b K(x,y) f(y) dy|^p dx$$

(By Holder inequality, where $1/p + 1/q = 1$)

$$\leq \int_a^b \Big(\int_a^b |K(x,y)|^q dy \Big)^{p/q} \int_a^b |f(y)|^p dy dx = \int_a^b \Big(\int_a^b |K(x,y)|^q dy \Big)^{p/q} \|f\|_p^p dx$$

(Since$\|f\|_p = 1$)

$$= \int_a^b \Big(\int_a^b |K(x,y)|^q dy \Big)^{p/q} dx.$$

Since $K(x,y)$ is continuous on $[a,b] \times [a,b]$ which is compact, there is $N \geq 0$ such that $|K(x,y)| \leq N$ for all $(x,y) \in [a,b] \times [a,b]$. Thus,

$$\int_a^b \Big(\int_a^b |K(x,y)|^q dy \Big)^{p/q} dx \leq \int_a^b \Big(\int_a^b M^q dy \Big)^{p/q} dx = \int_a^b (M^q (b-a))^{p/q} dx$$
$$= M^{1/p} (b-a)^{1+p/q}.$$

Thus T is bounded i.e., continuous with respect to $\| \cdot \|_p$. □

1.4 Generating Topology

1.4.8 Definition. Let X be a set with topologies σ and τ, then we say

(a) τ is **finer** than σ if $\tau \supset \sigma$, and

(b) τ is **coarser** than σ if $\tau \subset \sigma$.

1.4.9 Example. For any set X, the discrete topology $\tau = P(X)$ is the finest topology on X and the trivial topology $\tau = \{\emptyset, X\}$ is the coarsest topology on X.

1.4.10 Lemma. *If $\{\tau_i : i \in I\}$ are a collection of topologies on X, then $\tau = \bigcap_{i\in I} \tau_i$ is the finest topology that is coarser than each τ_i.*

Proof. First, we prove that τ satisfies all the axioms of topology. (i) Since $\emptyset$ and X are in τ_i for all $i \in I$, $\emptyset$ and X are also elements in $\tau = \bigcap_{i\in I} \tau_i$. (ii) Next, let U and V is open in (X, τ). Thus, $U, V \in \tau_i$ for each $i \in I$. Since τ_i is a topology for each $i \in I$, $U \cap V \in \tau_i$ for each $i \in I$. Thus, $U \cap V \in \bigcap_{i\in I} \tau_i = \tau$. (iii) Let $\{U_\alpha\}_{\alpha\in J}$ be a collection of open sets in (X, τ). Thus, $U_\alpha \in \tau_i$ for all $\alpha \in J$ and $i \in I$. Since, for each $i \in I$, τ_i is a topology, $\bigcup_{\alpha\in J} U_\alpha \in \tau_i$ for all $i \in I$. Thus, $\bigcup_{\alpha\in J} U_\alpha \in \bigcap_{i\in I} \tau_i = \tau$. In conclusion, by $(i), (ii)$ and (iii), τ is a topology on X. Now, we show that τ is the finest topology that is coarser than τ_i for every $i \in I$. Let σ be a topology on X that is coarser than τ_i for every $i \in I$. Let $U \in \sigma$. Since σ is coarser than τ_i for each $i \in I$, $U \in \tau_i$ for every $i \in I$. Thus, $U \in \bigcap_{i\in I} \tau_i = \tau$. Thus, $\sigma \subset \tau$. □

1.4.11 Definition. Let $\mathcal{A} \subset P(X)$ and $I = \{\sigma : \mathcal{A} \subset \sigma$ and σ is a topology on $X\}$, the set of all topologies containing $\mathcal{A}$. Define

$$< \mathcal{A} >:= \bigcap_{\sigma\in I} \sigma.$$

We call $< \mathcal{A} >$ the **topology generated by** $\mathcal{A}$ and we call $\mathcal{A}$ a generating set of the topology $< \mathcal{A} >$. If the union of all elements in $\mathcal{A}$ is X, it is called a **subbasis**. Thus, for any $\mathcal{A} \subset P(X)$, we can define $\mathcal{A}' = \mathcal{A} \cup \{X\}$. Then $\mathcal{A}'$ is a subbasis and both $\mathcal{A}$ and $\mathcal{A}'$ generate the same topology. Also, the set $\mathcal{A}$ is called a **basis** of a topology in X if each open set in X is a union of sets in $\mathcal{A}$.

1.4.12 Remark. Because $< \mathcal{A} >$ is the intersection of topologies, by the previous lemma, $< \mathcal{A} >$ is a topology on X, i.e., the above definition is well-defined. Also, $< \mathcal{A} >$ is the coarsest topology containing all sets in $\mathcal{A}$.

Proof. Let τ be a topology containing $\mathcal{A}$. Thus, $\tau \in \{\sigma : \mathcal{A} \subset \sigma,$ and σ is a topology $\} = I$. Thus, $< \mathcal{A} >= \bigcap_{\sigma\in I} \sigma \subset \tau$. Thus, $< \mathcal{A} >$ is coarser than any topology containing $\mathcal{A}$ i.e., it is the coarsest topology containing $\mathcal{A}$.

□

1.4.13 Examples. (i) The topology generated by empty set is the trivial topology, i.e., $< \emptyset >= \{\emptyset, X\}$ and the topology generated by $P(X)$ is discrete topology, i.e., $< P(X) >= P(X)$.

ii) Let $\mathcal{A} = \{B_r(x) \subset \mathbb{R}^n : x \in \mathbb{R}^n$ and $r > 0\}$, the collection of open balls in $\mathbb{R}^n$. Then $< \mathcal{A} >$ is the standard topology on $\mathbb{R}^n$.

1.4.14 Remark. Let $\mathcal{A}$ be a basis for a topology τ on X, i.e., each open set in τ is a union of elements in $\mathcal{A}$. Then $\tau =< \mathcal{A} >$.

Proof. Since $\mathcal{A}$ is a basis of τ, all elements in $\mathcal{A}$ are in τ. Since $< \mathcal{A} >$ is the coarsest topology containing $\mathcal{A}$, $< \mathcal{A} > \subset \tau$. Conversely, let $U \in \tau$. Then $U = \cup_{\alpha \in J} A_\alpha$ for some $A_\alpha \in \mathcal{A}, \alpha \in J$. Since $A_\alpha \in \mathcal{A}$ for all $\alpha \in J$ and $< \mathcal{A} >$ is a topology, by the axiom of topology, $U = \cup_{\alpha \in J} A_\alpha \in < \mathcal{A} >$. □

1.4.15 Remark. Let $\mathcal{A}$ is a subbasis of X. Define $\mathcal{B}$ be the collection of finite intersections of elements of $\mathcal{A}$. Let τ be the set of unions of elements in $\mathcal{B}$. Then τ is a topology on X, i.e., $\mathcal{B}$ is a basis for τ. Moreover, $\tau =< \mathcal{A} >$.

Proof. (i) Since $\emptyset$ is zero time intersection of elements in $\mathcal{A}$, $\emptyset \in \mathcal{B}$. Also, each element in $\mathcal{A}$ is one time intersection of elements in $\mathcal{A}$, thus it is in $\mathcal{B}$. Therefore, $\emptyset \in \tau$ and $\bigcup_{A \in \mathcal{A}} A = X \in \tau$. (ii) Let $U, V \in \tau$. Then, $U = \bigcup_{i \in I} B_i$ and $V = \bigcup_{j \in J} B_j$ where $B_i, B_j \in \mathcal{B}$ for all $i \in I$ and $j \in J$. Thus, $U \cap V = \bigcup_{i \in I, j \in J} (B_i \cap B_j)$. Since B_i and B_j are finite intersections of elements in $\mathcal{A}$. $B_i \cap B_j$ is also a finite intersection of elements in $\mathcal{A}$. (iii) Let $U_\alpha \in \tau$ for $\alpha \in I$. Then $U_\alpha = \bigcup_{i_\alpha \in I_\alpha} B_{i_\alpha}$. Then $\bigcup_{\alpha \in I} U_\alpha = \bigcup_{i \in \bigcup_{\alpha \in I} I_\alpha} B_i$. Since $B_i \in \mathcal{B}$, $\bigcup_{\alpha \in I} U_\alpha \in \tau$. In conclusion, τ is a topology. From the previos remark, we have that $\tau =< \mathcal{B} >$. We will show that $< \mathcal{A} >=< \mathcal{B} >$. Since $\mathcal{A} \subset \mathcal{B}$, $< \mathcal{A} > \subset < \mathcal{B} >$. Conversely, let $U \in < \mathcal{B} >$. Then, $U = \bigcup_{\alpha \in I} B_\alpha$ for $B_\alpha \in \mathcal{B}$ and $\alpha \in I$. Since B_α is a finite intersection of elements in $\mathcal{A}$ and $\mathcal{A} \subset < \mathcal{A} >$, by the axiom of topology that finite intersection of open set is open, $B_\alpha \in < \mathcal{A} >$. By the axom of topology that union of open sets is open, $U = \bigcup_{\alpha \in I} B_\alpha \in < \mathcal{A} >$. □

From both remarks, if $\mathcal{A}$ is a subbasis, an elements in $< \mathcal{A} >$ is a union of finite intersection of elements of $\mathcal{A}$. If $\mathcal{A}$ is not a subbasis, we have $\mathcal{A}' = \mathcal{A} \cup \{X\}$ and then $< \mathcal{A} >=< \mathcal{A}' >$. [1] on section 13.

1.4.16 Definition. Let X be a set and $(Y_i, \tau_i)_{i \in I}$ topological spaces.

(a) Given $f_i : X \to Y_i$ for each $i \in I$, then

$$\tau :=< f_i^{-1}(\tau_i) : i \in I >$$

is called **initial topology** associated with $(f_i, Y_i)_{i \in I}$.

(b) If $f_i : Y_i \to X$ for each $i \in I$, then

$$\tau := \{U \subset X : f_i^{-1}(U) \in \tau_i \text{ for all } i \in I\}$$

is called the **final topology** associated with $(f_i, Y_i)_{i \in I}$.

1.4.17 Remark. Let $\sigma_i = \{f_i^{-1}(U) : U \in \tau_i\}$. Because f_i^{-1} is compatible with unions and intersections, and τ_i is a topology on Y_i for each $i \in I$, we can prove that $\sigma_i = \{f_i^{-1}(U) : U \in \tau_i\}$ is a topology on X for every $i \in I$. But the union of all σ_i might not be a topology. In addition,

[1] For more information about topology basis and subbasis, check out this book: Munkres, J. R. (2000). Topology 2nd. Upper Saddle River, NJ: Prentice Hall, Inc.

the initial topology is the topology generated by $\mathcal{A} = \bigcup_{i \in I}\{f_i^{-1}(U) : U \in \tau_i\} = \bigcup_{i \in I} \sigma_i$, i.e., the initial topology is

$$\tau =< \bigcup_{i \in I} \sigma_i > .$$

Thus, it is the topology.

In addition, $\sigma_i = \{U \subset X : f_i^{-1}(U) \in \tau_i\}$. By the properties of f^{-1} which is compatible with unions and intersections, σ_i is a topology for every $i \in I$. Moreover, the final topology can be alternatively written as

$$\tau := \{U \subset X : f_i^{-1}(U) \in \tau_i \text{ for all } i \in I\} = \bigcap_{i \in I} \sigma_i.$$

By the Lemma 1.4.10, the final topology is also a topology. In conclusion, the definition of the initial topology and the final topology are well-defined.

1.4.18 Lemma. *The initial topology τ on X associated with $\{f_i : X \to Y_i\}$ is the coarsest topology with respect to which all f_i are continuous. If Z is a topological space, then $h : Z \to X$ is continuous if and only if $f_i \circ h$ is continuous for each $i \in I$.*

Proof. Denote $\sigma_i = \{f_i^{-1}(U) : U \in \tau_i\}$. (i) First, we show that $f_i : X \to Y_i$ is continuous for every $i \in I$. Let $U \in \tau_i$. Then $f_i^{-1}(U) \in \sigma_i \subset \bigcup_{i \in I} \sigma_i$. The initial topology is the topology generated by $\bigcup_{i \in I} \sigma_i$. Thus, it contains all elements in $\bigcup_{i \in I} \sigma_i$. Therefore, $f_i^{-1}(U)$ is open in the initial topology, i.e., $f_i : X \to Y_i$ is continuous for every $i \in I$.

(ii) Next, we show that τ is the coarsest topology which all f_i are continuous. Let σ be a topology on X such that $f_i : X \to Y_i$ is continuous for every $i \in I$. Let U be open set in Y_i. Then, $f_i^{-1}(U) \in \sigma$. Thus σ is a topology containing $\bigcup_{i \in I} \sigma_i$. Since the initial topology is generated by $\bigcup_{i \in I} \sigma_i$ which is the coarsest topology containing $\bigcup_{i \in I} \sigma_i$, $< \bigcup_{i \in I} \sigma_i > \subset \sigma$.

(iii) Now, we show the last statement. Let $h : Z \to X$ is continuous. Then $f_i \circ h : Z \to Y_i$ is also continuous because it is composed by two continuous functions. Conversely, assume that $h : Z \to X$ is a map and $f_i \circ h : X \to Y_i$ is continuous for every $i \in I$. Let U be open set in the initial topology on X. Thus U is a union of finite intersection of elements in $\bigcup_{i \in I} \sigma_i$. Thus, $U = \bigcup_{\alpha \in J} \bigcap_{j=1}^{j_\alpha} A_j^\alpha$ where $A_j^\alpha \in \bigcup_{i \in I} \sigma_i$ for all $\alpha \in J$ and $j = 1, ..., j_\alpha$. Then, $h^{-1}(U) = \bigcup_{\alpha \in J} \bigcap_{j=1}^{j_\alpha} h^{-1}(A_j^\alpha)$. Since $A_j^\alpha \in \bigcup \sigma_i$, $A_j^\alpha = f_i^{-1}(V_i)$ for some $i \in I$ and $V_i \in \tau_i$. Thus, $h^{-1}(A_j^\alpha) = h^{-1} \circ f_i^{-1}(V_i) = (f \circ h)^{-1}(V_i)$. Since $f_i \circ h$ is continuous, $h^{-1}(A_j^\alpha)$ is open in Z for each α and $j = 1, ..., j_\alpha$. Thus, $h^{-1}(U)$ which is the union of finite intersection of $h^{-1}(A_j^\alpha)$ is open by the axiom of topology. □

1.4.19 Lemma. *The final topology τ on X associated with $\{f_i : Y_i \to X : i \in I\}$ is the finest topology for which all f_i are continuous. If Z is a topological space, then a map $h : X \to Z$ is continuous if and only if $h \circ f_i$ is continuous for each $i \in I$.*

Proof. Denote $\sigma_i = \{U \subset X : f_i^{-1}(U) \in \tau_i\}$. (i) First, we show that $f_i : Y_i \to X$ is continuous for every $i \in I$. Let U be open set in X. Then, $f_i^{-1}(U) \in \tau_i$ for all $i \in I$, i.e., $f_i : Y_i \to X$ is continuous.

(ii) Next, we show that the final topology is the finest topology for which all f_i is continuous for every $i \in I$. Let σ is a topology on X which $f_i : Y_i \to X$ is continuous. Let $U \in \sigma$.

Then $f_i^{-1}(U) \in \tau_i$ for every $i \in I$. Thus, $U \in \{U \subset X : f_i^{-1}(U) \in \tau_i\}$ for every $i \in I$, i.e., $U \in \bigcap_{i \in I} \sigma_i$. Thus, $\sigma \subset \tau$, i.e., the final topology is finer that σ.

(iii) Assume that h is continuous. Since the composition of continuous functions is continuous, $h \circ f_i : Y_i \to Z$. is also continuous for every $i \in I$. Conversely, assume that $h : X \to Z$ be a map and $h \circ f_i : Y_i \to Z$ is continuous. Let U be open set in Z. Then $(h \circ f_i)^{-1}(U) = f^{-1} \circ h^{-1}(U)$ is open in Y_i for each $i \in I$. Thus, $h^{-1}(U) \in \bigcap_{i \in I} \sigma_i$. Thus, by the definition of the final topology, $h^{-1}(U)$ is open in the final topology. Hence h is continuous. $\square$