Functional Analysis, Math 7320
Lecture Notes from September 08, 2016

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Last time we saw two ways to generate new topologies from old ones. We saw that the initial topology is the coarsest one, under some conditions, and the final topology is the finest one, in another specific context. Topologies resulting from these constructions occur rather frequently, although the terms "initial" and "final" are seldom mentioned.

1.3.20 Examples. (a) Given an equivalence relation $\sim$, on the topological space $\mathcal{X}$, and a quotient space $Y := \mathcal{X}/\sim$, the canonical projection[1] $q : x \mapsto [x]$, defines a *final* topology on Y called the *quotient topology*. Given any subset $U \subset Y$, U is open in the quotient topology if and only if $q^{-1}(U)$ is open in $\mathcal{X}$.

A more specific example is a semimetric space $(\mathcal{X}, d)$ on which we identify points that are at a null distance from each other. Formally, an equivalence relation $\sim$ is defined by: $x \sim y \Leftrightarrow d(x, y) = 0$.

Facts.

- The function $\tilde{d}\left([x], [y]\right) := d(x, y)$ is a well-defined metric on $\mathcal{X}/\sim$.
- An ϵ-ball in the quotient space $\mathcal{X}/\sim$, lifts to an ϵ-ball in the semimetric space $\mathcal{X}$, via the inverse of the canonical projection $q : x \mapsto [x]$; and ϵ-balls in $\mathcal{X}$ project to ϵ-balls in $\mathcal{X}/\sim$ via q.
- The metric $\tilde{d}$ induces the quotient topology on $\mathcal{X}/\sim$.

Proof.

- If $x_1 \sim x_2$ and $y_1 \sim y_2$ are four points in $\mathcal{X}$, a simple use of the triangle inequality and the symmetry of the semimetric yields $d(x_1, y_1) = d(x_2, y_2)$. So $\tilde{d}$ is well-defined and inherits all the properties of a semimetric from d. Finally, for any $x, y \in \mathcal{X}$, if $\tilde{d}([x], [y]) = 0$, then $d(x, y) = 0 \Rightarrow x \sim y \Rightarrow [x] = [y]$. So $\tilde{d}$ is a metric.
- Since for every $x, y \in \mathcal{X}$, $d(x, y) = \tilde{d}([x], [y])$, it comes that for any $\epsilon > 0$, $d(x, y) < \epsilon \Leftrightarrow \tilde{d}([x], [y]) < \epsilon$. Hence, $q^{-1}\left(B_\epsilon^{\tilde{d}}([x])\right) = B_\epsilon^d(x)$ and $q\left(B_\epsilon^d(x)\right) = B_\epsilon^{\tilde{d}}([x])$, for all $x \in \mathcal{X}$.
- Call $\tilde{\mathcal{B}}$, the collection of ϵ-balls in the quotient space $\mathcal{X}/\sim$. We already know that it is a basis for the topology induced by $\tilde{d}$, we will show that it is also a basis for the quotient topology. Let $\tilde{U}$ be an open set in $\mathcal{X}/\sim$, with respect to the quotient topology, and let $[x] \in \tilde{U}$. It is enough to show that $\exists \tilde{B} \in \tilde{\mathcal{B}}$ such that $[x] \in \tilde{B} \subset U$.

[1] The notation $[x]$ refers to the equivalence class of x.

By definition, $U := q^{-1}\left(\tilde{U}\right)$ is an open set in $\mathcal{X}$. In particular, it can be expressed as a union of basis elements, that is, as a union of ϵ-balls in $\mathcal{X}$: $U = \bigcup_{i\in I} B_i$. The point x (some representative of $[x]$) *must* lie in one of these ϵ-balls, say $x \in B_j$ for a fixed $j \in I$. Using the previous bullet point, we conclude that $[x] \in q(B_j) \subset \tilde{U}$.

□

The previous facts tell us that one may construct a topology on $\mathcal{X}/\sim$ in two equivalent ways:

1. first generate the topology from the semimetric on $(\mathcal{X}, d)$, and then construct the *quotient* topology on $\mathcal{X}/\sim$;
2. first *specialize* the semimetric to a *metric* on $\mathcal{X}/\sim$, and then generate the topology from this metric.

As an even more specific case of the current example, recall how L^2 is constructed as a quotient space under the equivalence relation $f \sim g \Leftrightarrow f = g$ a.e.. In this process, the semimetric induced by the seminorm on the initial function space becomes a metric (induced by the norm) in L^2.

This was an example of final topology. The next example will be a typical occurrence of initial topology in the context of cartesian products.

(b) Let $(\mathcal{X}_i)_{i\in I}$ be a family of topological spaces[2] and define their cartesian product as follows:

$$\mathcal{X} = \prod_{i\in I} \mathcal{X}_i := \left\{ x : I \to \bigcup_{i\in I} \mathcal{X}_i \;\middle|\; x(i) \in \mathcal{X}_i, \forall i \in I \right\}.$$

We denote an element of $\mathcal{X}$ by $x = (x_i)_{i\in I}$, where $x_i := x(i)$ for each $i \in I$. Furthermore, for each $i \in I$, the *projection onto* $\mathcal{X}_i$ is defined by: $\pi_i : \mathcal{X} \to \mathcal{X}_i, (x_j)_{j\in I} \mapsto x_i$. The *initial* topology associated with these $(\pi_i)_{i\in I}$ is called the *product topology* on $\mathcal{X}$.

We will encounter this topology later in the context of the weak-$*$ topology.

(c) This is an illustration of the previous example. Let $\mathcal{X}$ be a set and Y a topological space. Denote by $F(\mathcal{X}, Y)$ the set of all maps $f : \mathcal{X} \to Y$. Interpreting $F(\mathcal{X}, Y)$ as $Y^{\mathcal{X}}$, then the product topology on $F(\mathcal{X}, Y)$ is the coarsest topology such that for each fixed point $x \in \mathcal{X}$, the point evaluation map $\delta_x : F(\mathcal{X}, Y) \to Y, f \mapsto f(x)$, is continuous.

We will also see this again later.

(d) If $(\mathcal{X}, \tau)$ is a topological space and $Y \subset \mathcal{X}$, then,

$$\tau_Y := \{U \cap Y \;:\; U \in \tau\},$$

defines a topology on Y called the *trace topology*[3]. This is an *initial* topology with respect to the inclusion map $i : Y \to \mathcal{X}, y \mapsto y$.

[2] A note about the notation: when writing $(\mathcal{X}_i)_{i\in I}$ we imply that a specific map $i \mapsto \mathcal{X}_i$ is fixed, as opposed to the notation $\{\mathcal{X}_i\}_{i\in I}$, which merely represents a set (the order doesn't matter and elements are not repeated).

[3] It is also commonly called the subspace topology, but we shall avoid this expression in this class in order to avoid confusion with stronger versions of the word subspace, such as for linear subspaces.

1.3.21 Remark. There is a natural decomposition of a map related to the *quotient* and *trace* topologies. Given two topological spaces $\mathcal{Z}$, $\mathcal{X}$ and a map $f : \mathcal{Z} \to \mathcal{X}$, we may use the inclusion map $i : f(\mathcal{Z}) \to \mathcal{X}$ to define the trace topology on $f(\mathcal{Z})$. We can then characterize the continuity of f as follows:

1.3.22 Claim. The map f is continuous if and only if $\tilde{f} : \mathcal{Z} \to f(\mathcal{Z}), x \mapsto f(x)$, is continuous (using the trace topology on the target space).

Proof. If $\tilde{f}$ is continuous, then $f = i \circ \tilde{f}$ is continuous as a composition of continuous functions. Conversely, assume that f is continuous and let V be open in $f(\mathcal{Z})$. Then there is an open U in $\mathcal{X}$ such that $V = U \cap f(\mathcal{Z})$. We then have,

$$\tilde{f}^{-1}(V) = f^{-1}(U \cap f(\mathcal{Z})) = f^{-1}(U),$$

which is open by continuity of f. $\square$

More generally, given any $f : \mathcal{Z} \to \mathcal{X}$, we may define an equivalence relation on $\mathcal{Z}$ by, $x \sim y \Leftrightarrow f(x) = f(y)$, and form the associated quotient space[4] $\mathcal{Z}/\sim$, endowed with the quotient topology. Then the map, $\bar{f} : \mathcal{Z}/\sim \to f(\mathcal{Z}), [x] \mapsto f(x)$, is a (well-defined) bijection[5] and the following diagram commutes (recall that we equipped $f(\mathcal{Z})$ with the trace topology):

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, "\tilde{f}"] & \mathcal{X} \\
\mathcal{Z}/\sim \arrow[r, "\bar{f}"] & f(\mathcal{Z}) \arrow[u, "i"']
\end{tikzcd}$$

Furthermore, we have the following characterization of the continuity of f:

1.3.23 Claim. The map f is continuous $\Leftrightarrow$ $\bar{f}$ is.

Proof. It is enough to show that $\tilde{f}$ is continuous if and only if $\bar{f}$ is. We already know that $\tilde{f} = \bar{f} \circ q$, therefore $\bar{f}$ continuous $\Rightarrow$ $\tilde{f}$ continuous. For the converse, assume that $\tilde{f}$ is continuous. We first note that for any $x \in \mathcal{Z}$, $\tilde{f}(\{y \in \mathcal{Z} \ : \ [y] = [x]\})$ is a singleton in $f(\mathcal{Z})$. In other words, $\tilde{f}$ is *constant on the fibers* of q. Next, consider an open set V in $f(\mathcal{Z})$. Then, $\tilde{f}^{-1}(V) = q^{-1}\left(\bar{f}^{-1}(V)\right)$, is open by our assumption. But $\mathcal{Z}/\sim$ being endowed with the quotient topology induced by q implies that $\bar{f}^{-1}(V)$ is open in $\mathcal{Z}/\sim$. We are done. $\square$

In many instances, we are used to characterizing a topology on a set by the convergence of its sequences. In most generality, this approach is "not enough".

1.3.24 Remark. The product topology can show a more general behavior than metric spaces. If $(\mathcal{X}, d)$ is a metric space, Y a subset of it, and x a point in Y, then there is a sequence $(y_n)_{n\in\mathbb{N}}$ contained in Y such that $\lim_{n\to\infty} y_n = x$ (where the limit is understood with respect to d).

[4] We denote the canonical projection $q : x \mapsto [x]$.

[5] It is well-defined because if $x \sim y$ in $\mathcal{Z}$, then $f(x) = f(y)$ by construction. It is surjective because, given any $f(x) \in f(\mathcal{Z})$, $\bar{f}([x]) = f(x)$, and it is injective because $\bar{f}([x]) = \bar{f}([y]) \Rightarrow f(x) = f(y) \Rightarrow x \sim y \Rightarrow [x] = [y]$.

(a) In contrast to this, define the set of all *binary* functions on $\mathbb{R}$, $\mathcal{X} := F(\mathbb{R}, \{0,1\})$, where $\{0,1\}$ is endowed with the discrete topology and $\mathcal{X}$ with the product topology[6]. Consider the set of all functions which assume the value 1, at most finitely many times:

$$Y := \{f \in \mathcal{X} \ : \ |\{t : f(t) = 1\}| < \infty\}.$$

1.3.25 Claim. The constant function $\mathbb{1} : t \mapsto 1$ *is in* $\bar{Y}$.

Proof. To prove this claim we need to show that every neighborhood of $\mathbb{1}$ intersects Y non-trivially. We may even simplify the strategy further. It is enough to show that any basis element in $\mathcal{X}$ containing $\mathbb{1}$ has a non-trivial intersection with Y. Recall that in the product topology at hand, a basis element can always be expressed as a finite intersection of elements from a subbasis. And, the subbasis that we consider is $\left\{\pi_t^{-1}\{U\} \ : \ U \subset \{0,1\} \text{ is open and } t \in \mathbb{R}\right\}$. Thus, if U is a basis element of $\mathcal{X}$ containing $\mathbb{1}$, then there are $t_1, \ldots, t_n \in \mathbb{R}$ such that,

$$U = \bigcap_{j=1}^{n} \pi_{t_j}^{-1}(\{1\}) = \{f \in \mathcal{X} \ : \ f(t_j) = 1 \text{ for } j \in \{1, \ldots, n\}\}.$$

Note that the function γ which is 0 everywhere, except exactly on $\{t_1, \ldots, t_n\}$, is an element of $U \cap Y$. Hence, $U \cap Y \neq \emptyset$ and the claim is proved. $\square$

(b) However, there is no sequence $(f_n)_{n\in\mathbb{N}}$ in Y with $\lim_{n\to\infty} f_n = \mathbb{1}$. This is because the following set,

$$E := \{t \in \mathbb{R} \ : \ \exists n \in \mathbb{N} \text{ such that } f_n(t) \neq 0\},$$

is countable (and therefore $E^c \neq \emptyset$). For $t \notin E$, $\lim_{n\to\infty} f_n(t) = 0$.

Moral To retain the equivalence between a topology and the notion of convergence, we need a concept "more general" than limits of sequences.

1.4 Convergence

We recall axioms of ordering below.

1.4.26 Definition. Let I be a nonempty set. A binary relation $\leq$ on I is called an *order(ing)* (or *quasi-order(ing)*) and $(I, \leq)$ an *ordered* (or *quasi-ordered*) *set* if:

(1) for all $a \in I$, $a \leq a$ (reflexivity)

(2) for all $a, b, c \in I$, $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitivity)

If we have, in addition to (1) and (2), that,

(3) $a \leq b$ and $b \leq a \Rightarrow a = b$, for all pairs $(a, b) \in I^2$, (antisymmetry)

then the set is *partially ordered*. If in addition to (1), (2) and (3), the following holds,

[6] Note that we may interpret $\mathcal{X} = \{0,1\}^{\mathbb{R}}$.

(4) for all $a, b \in I$, either $b \leq a$ or $a \leq b$, (totality)

then the relation $\leq$ is called a *linear* (or *total*) order. If in addition to (1) and (2), we have that,

(5) for all $a, b \in I$, there exists $c \in I$ such that both, $a \leq c$ and $b \leq c$,

then $(I, \leq)$ is called a *directed set*.

1.4.27 Examples. (a) The set $(\mathbb{N}, \leq)$ is linearly ordered.

(b) Each linearly ordered set is directed.

(c) If $(\mathcal{X}, \tau)$ is a toplogical space and $x \in \mathcal{X}$, then the collection of neighborhoods of x, $(\mathcal{U}(x), \supseteq)$ is directed. Note the somewhat counter-intuitive use of $\leq$, we have, $U_1 \leq U_2 \Leftrightarrow U_2 \subseteq U_1$.

(d) If $\mathcal{X}$ is any set and $I = \mathcal{P}(\mathcal{X})$ its power set, then $(I, \subseteq)$ is directed.

We are now ready to generalize to sequences.

1.4.28 Definition. Let $(I, \leq)$ be a directed set and $\mathcal{X}$ a set. A map $x : I \to \mathcal{X}$ is called a *net* in $\mathcal{X}$. We also write $(x_i)_{i \in I}$.

1.4.29 Remark. Every sequence is a net. However, as opposed to sequences, nets do not have to be linearly ordered.

1.4.30 Definition. If $x : I \to \mathcal{X}$ is a net and $f : \mathcal{X} \to Y$ is any map, then $f \circ x : I \to Y$ is a net on Y called the *image net* of x under f.

1.4.31 Definition. (a) Let $(\mathcal{X}, \tau)$ be a topological space and $a \in \mathcal{X}$ a point of it. We say that a net $(x_i)_{i \in I}$ *converges* to a if for each $U \in \mathcal{U}(a)$ there is an $n_0 \in I$ such that, for all $i \geq n_0$ we have, $x_i \in U$. In this case, we write $x_i \to a$ and call a the *limit* of $x = (x_i)_{i \in I}$.

(b) Let's look at an example that will be very important to us in the following. If $\mathcal{X}$ is a vector space equipped with a topology τ, J a set and $y : J \to \mathcal{X}$, $j \mapsto y_j$, a map, then we consider the directed set,

$$I := \{K \subseteq J \ : \ |K| < \infty\} ,$$

with partial order induced by $\subseteq$. Further define $x : I \to \mathcal{X}$ and $x_F = \sum_{j \in F} y_j$ (note that the sum is finite). If $(x_F)_{F \in I}$ converges, then we write $\sum_{j \in J} y_j := \lim_{F \in I} x_F$ and say that $\sum_{j \in J} y_j$ *converges*. This should remind you of the notion of *unconditional convergence*. Indeed, for $\mathcal{X} = \mathbb{K}$, $J = \mathbb{N}$, this is equivalent to absolute convergence, because the limit is required to be independent of the order of summation, and therefore by Riemann's summability theorem, this characterizes absolute convergence.

Next time, we will be looking at nets and closures, and at the notion of *filter base*.