Functional Analysis, Math 7320
Lecture Notes from September 13, 2016

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Last Time

- Ordering
- Net and Convergence

We have seen from the previous lecture that a function f is continuous at a point a if and only iff a convergent net $(x_i)_{i\in I}$ converges to a, $f(x_i)$ converges to $f(a)$. In this lecture note, we introduce a concept of filterbase. Then, we will see that convergence of nets is related with convergence of filterbases. Also, a continuity of a function can be characterized by convergence of filterbases.

Filterbases

1.4.32 Definition. Let X be a set. A set $\mathbb{B} \subset P(X) \setminus \{\emptyset\}$ is called a filterbase if $(\mathbb{B}, \supseteq)$ is a directed set. A filterbase is called a filter if $A \subset B$ and $A \in \mathbb{B}$ implies $B \in \mathbb{B}$. If $\mathbb{B}$ is a filterbase then $\hat{\mathbb{B}} = \{B \subset X : \exists A \in \mathbb{B}, A \subset B\}$ is the filter generated by $\mathbb{B}$.

1.4.33 Remark. Being a directed set means that two elements in a filterbase always intersect. In the other words, if $\mathbb{B}$ is a filterbase and $A, B \in \mathbb{B}$, $A \cap B \neq \emptyset$.

1.4.34 Examples. (a) If (X, τ) is a topological space then the set of all neighborhoods of x:

$$\mathcal{U}(x) = \{B \subset X : x \in U \subset B \text{ for some open set } U \text{ containing } x\}$$

with "$\supseteq$" is a filter called the neighborhood filter of X.

(b) If $(x_i)_{i\in I}$ is a net then $\mathbb{B} = \{B_i : i \in I\}$ with $B_i = \{x_j : j \geq i\}$ is called the filterbase associated with the net $(x_i)_{i\in I}$.

(c) $\mathbb{B} = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$ is a filterbase in $\mathbb{R}$.

Characterization of Topologies by Convergence of Filterbases or Nets

1.4.35 Definition. let (X, τ) be a topological space and $x \in X$. We say that a filterbase $\mathbb{B}$ converges to x (denote $\mathbb{B} \to x$), if for each $U \in \mathcal{U}(x)$ there is $B \in \mathbb{B}$ such that $B \subset U$. This condition is equivalent to $\mathcal{U}(x) \subset \hat{\mathbb{B}}$.

1.4.36 Example. A filterbase $\mathbb{B} = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$ converges to 0. Note that each element in this filterbase does not contain 0 but $\mathbb{B}$ converge to 0.

Proof. Let $U \in \mathcal{U}(0)$. There exists $(-\epsilon, \epsilon) \subseteq U$, for some $\epsilon > 0$. By archimedian property , there is $n \in \mathbb{N}$ such that $1/n < \epsilon$. So, $(0, 1/n) \subseteq (-\epsilon, \epsilon) \subseteq U$. Thus the filterbase $\mathbb{B}$ converges to 0. □

To formulate a characterization of convergence ,we use the axiom of choice.
Axiom: If $(X_i)_{i \in I}$ is a family of non empty sets then there is map $x : I \to \bigcup_{i \in \mathbb{I}} X_i$ with $x(i) \in X_i$ for every $i \in I$.

1.4.37 Lemma. *A filterbase $\mathbb{B}$ on a topological space X converges to $y \in X$ if and only if each net with $I = \mathbb{B}$ and $x_B \in B$ for each $B \in \mathbb{B}$ converges to y.*

Proof. If $\mathbb{B}$ converges to y, then by definition: for each $U \in \mathcal{U}(y)$ there is $B \in \mathbb{B}$ with $B \subset U$ but then if $(x_B)_{B \in \mathbb{B}}$ for each $B \in \mathbb{B}$ we have for each $B' \subset B$, $x_{B'} \in B' \subset B \subset U$. Thus, $(x_B)_{B \in \mathbb{B}}$ converges to y.
Conversely, assume $\mathbb{B}$ does not converge to y . Then ,there exist $U \in \mathcal{U}(x)$ such that for no $B \in \mathbb{B}$,$B \cap U^c = \emptyset$ this means for any $B \in \mathbb{B}$, there is $x_B \in B$ with $x_B \notin U$, thus $(x_B)_{B \in \mathbb{B}}$ does not converge to y. □

1.4.38 Remark. If I is countable , i.e. $I \subset \mathbb{N}$ then the elements of $\prod_{i \in I} X_i$ can be constructed by induction.With the added axiom , we can admit uncountable I

1.4.39 Theorem. *Let (X, τ) be a topological space, $Y \subset X$ and $a \in X$. Then $TFAE$*

(a) $a \in \bar{Y}$.

(b) there is a net $(x_i)_{i \in I}$ in Y with $\lim_{i \in I} x_i = a$.

(c) there is a filterbase $\mathbb{B}$ in Y converging to a.

Proof. We first show $(a) \Leftrightarrow (b)$. Assume $a \in \bar{Y}$, then for each $U \in \mathcal{U}(a)$, there is $x_u \in U \cap Y \neq \emptyset$ and $a \in U$. Hence $x_u \to a$. Conversely, assume $a \notin \bar{Y}$ then $a \in X \setminus \bar{Y} = V$. Then by openness of V, there is $U \in \mathcal{U}(a)$ with $U \cap Y = \emptyset$
If I indexes a net $x_i \to a$ then we know there is i_0 such that for all $i \geq i_0$, we have $x_i \in U$, so $x_i \notin Y$. Hence , there is no such net in Y converging to a .
Next , we show $(b) \Leftrightarrow (c)$
let $(x_i)_{i \in I}$ be a net in Y with $x_i \to a$.Then , $\mathbb{B} = \{B_i : i \in I\}$ with $B_i = \{x_j : i \leq j\}$ gives a filterbase with $\mathbb{B} \to a$.Finally , if $\mathbb{B}$ is a filterbase in Y with $\mathbb{B} \to a$ then if $I = \mathbb{B}$, and we choose $x_B \in B$ for each $B \in \mathbb{B}$ .Then $x_B \to a$. □

Characterization of Continuity with Nets or Filterbases We can use nets and filterbases to characterize continuity

1.4.40 Theorem. *let $f : X \to Y$ with topological spaces X, Y and $a \in X$. Then $TFAE$*

(1) f is continuous at a.

(2) *For each net* $f(x_i) \to f(a)$.

(3) *For each filterbase* $\mathbb{B}$ *in* X *with* $\mathbb{B} \to a$ *, we have that* $f(\mathbb{B}) = \{f(B) : B \in \mathbb{B}\}$ *converges to* $f(a)$.

Proof. We show $(1) \Leftrightarrow (2)$. Assuming f is continuous at a and given $V \in \mathcal{U}(f(a))$,applying f to both side Then $U = f^{-1}(V)$ is in $\mathcal{U}(a)$, so if $x_i \to a$, there is i_0 such that for all $j \geq i_0$, $x_j \in U$ and hence $f(x_j) \in f(U) = V$.This means the image net converges to $f(a)$. Next,assume f is not continuous at a, then there is $V \in \mathcal{U}(f(a))$ and for each $U \in \mathcal{U}(a)$, there is x_u with $x_u \in U$ but $(f(x_u)) \notin V$. Thus, we get $x_u \to a$ but $(f(x_u))_{u \in \mathcal{U}(a)}$ does not converge to $f(a)$.
Next, we show $(2) \Leftrightarrow (3)$. For any filterbase $\mathbb{B}$, we can define a net (x_B) such that $x_B \in B$ for each $B \in \mathbb{B}$. If $\mathbb{B} \to a$, then $x_B \to a$. On the other hand, if (x_i) for $i \in I$ is a net with a direted set I. Define $B_i = \{x_j : j \geq i\}$. Then, $\mathbb{B} = \{B_i : i \in I\}$ will be a filterbase. If $x_i \to a$, then $\mathbb{B} \to a$. By using these relations, it follows that $(2) \Leftrightarrow (3)$. □

1.4.41 Remarks. (1) If τ and σ are topologies on X, then τ is finer than σ if and only if $Id : (X, \tau) \to (X, \sigma)$, $x \mapsto x$ is continuous and $\sigma = \tau$ is equivalence to Id being a homeomorphism.

(2) The characterization of continuity gives us that τ is finer than σ if all nets /filterbases that converge with respect to τ also converge with respect to σ. Thus, the notion of convergence of nets/filterbases characterizes topologies.

Proof. (1) We know that $Id(A) = A$ for any $A \subset X$ and $Id^{-1}(B) = B$ for any $B \subset X$. Let U is open set in (X, σ). Since Id is continuous, $Id^{-1}(U) = U$ is also open in (X, τ). Thus, $U \in \tau$. Therefore, $\sigma \subset \tau$ i.e., τ is finer that σ. Conversely, if τ is finer than σ, $Id^{-1}(U) \in \tau$ for any $U \in \sigma$. Thus, Id is continuous. Therefore, If $\sigma = \tau$, both Id and Id^{-1} are continuous. Thus, Id is homeomorphism.

(2) If (X, τ) and (X, σ) are homeomorphism, the identity map $Id : X \to Y$ is a homeomorphism. Thus, by the previous theorem, any net (x_i) or any filterbase $\mathbb{B}$ converging to a, then $Id(x_i)$ or $Id(\mathbb{B})$ converge to $Id(a)$. Also, this is true for Id^{-1}. Conversely, if for any net (x_i) or any filterbase $\mathbb{B}$ converging to a, $Id(x_i)$ or $Id(\mathbb{B})$ converge to $Id(a)$ and also if this is true for Id^{-1}, by the previous theorem, Id and Id^{-1} is continuous. Thus, Id is homeomorphism i.e., $\sigma = \tau$. This shows convergence of filterbases and nets characterizes topologies on X. □

1.4.42 Corollary. *Let* X *be a set with th initial topology induced by* $f_i : X \to Y_i$. *Then*

(a) *A filterbase* $\mathbb{B}$ *in* X *converges to* $a \in X$ *if and only if* $f_i(\mathbb{B})$ *converges to* $f_i(a)$ *for each* $i \in X$.

(b) *A net* $(x_j)_{j \in J}$ *in* X *converges to* $a \in X$ *iff each image net* $f_i(x_j)$ *converges to* $f_i(a)$ *for all* $i \in I$.

Proof. By previous lectures, we know that, for every i, $f_i : X \to Y_i$ is continuous with respect to the initial topology induced by $f_i : X \to Y_i$. Thus, by the above theorem, a filterbase $\mathbb{B}$ converges to a if and only if $f(\mathbb{B})$ converges to $f_i(a)$. Also, by the above theorem, we also have that a net (x_i) converges to a if and only if $f(x_i)$ converges to $f_i(a)$. □

we studied nets but an interesting question rises why we really need nets and for which spaces, continuous sequences are enough to characterize the topology. So we will introduce the new topic, countability.

Countability

1.4.43 Definition. Let (X, τ) be a topological space.Then

(a) If $x \in X$ we call $\mathbb{B} \subset \mathcal{U}(x)$ is a neighborhood (or local) basis if for each $U \in \mathcal{U}(x)$, there is $B \in \mathbb{B}$ with $B \subset U$.

(b) We say that X is first countable if for each $x \in X$ there is a countable neighborhood basis.

(c) We say X is a second countable set if τ is generated by a countable set(in that case τ has a countable basis).

(d) X is separable if it has a countable dense set

1.4.44 Examples. (a) $(\mathbb{K}^n, \|.\|)$ is separable.

(b) l^p, $1 \leq p < \infty$ is separable.

(c) l^∞ is not separable .

(d) c_0 is separable.

Proof. (a) For $\mathbb{K} = \mathbb{R}$, $\mathbb{Q}^n$ is a countable set and it is dense in $\mathbb{R}^n$. If $\mathbb{K} = \mathbb{C}$, an countable dense set in $\mathbb{K}$ will be $(\mathbb{Q} \times i\mathbb{Q})^n$.

(b) To prove l^p, $1 \leq p < \infty$, is separable, we show that it contains a countable dense subset. Let

$$A_n = \{(b_0, b_1, ..., b_n, 0, ...) : b_i \in \mathbb{Q} \text{ for } i = 0, ..., n\} \text{ and,}$$

$$A = \bigcup_{n=0}^{\infty} A_n.$$

Because a sequence of the form $(b_0, b_1, ..., b_n, 0, ...)$ has only finitely many terms of non-zeros, $\sum_{i=0}^{\infty} |b_i|^p = \sum_{i=0}^{n} |b_i|^p < \infty$. Thus, these sequences are in l^p. Since $\mathbb{Q}$ is countable and finite product of countable sets is countable and then the union of countable sets is countable ,so A is countable. Now we want to prove A is dense in l^p. Given any $x = (x_n) \in l^p$, and for $\epsilon > 0$ there is $y \in A$ such that $d(x, y) < \epsilon$. Then we have

$$\sum_{n=0}^{\infty} |x_n|^p < \infty$$

Thus, given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\sum_{n=m+1}^{\infty} |x_n|^p < \epsilon.$$

Now for $0 \leq i \leq m$, choose $b_i \in \mathbb{Q}$ such that $|b_i - x_i| < \epsilon/m$. Then the element $y = (b_0, b_1, ..., b_m, 0, 0, ...) \in A$.

(c) Let $x : S = \{0,1\}^{\mathbb{N}} \to (x_n)_{n\in\mathbb{N}}$, $x_n = (-1)^n$ and $V_x = B_{1/3}(X)$ then if $x \neq y$ then $V_x \cap V_y = \emptyset$. If $(c_m)_{m\in\mathbb{N}}$ is dense in l^∞ then for each x there would be $m(x)$ such that $m(x) \in V_x$. Choosing for each x the smallest m such that $x = m(x)$ would give 1-1 map from S to $\mathbb{N}$, but this cannot exist by cardinality of S.

□