# Functional Analysis, Math 7320
# Lecture Notes from September 15, 2016

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*1.4.44 Examples* (Separable or Non-Separable topological spaces). (d) $c_0$ is separable.

*Proof.* We already know that $(\mathbb{K}^n, \|.\|_\infty)$ is separable. Defining the embedding $S_n = i(\mathbb{K}^n)$, where $i(x) = (x_1, x_2, ..., x_n, 0, 0, ...)$, we see that $\bigcup_{n\in\mathbb{N}} S_n$ is separable. Now, letting $x \in c_0$ and $\epsilon > 0$, by definition of $c_0$, we can find $n$ such that for all $m \geq n$ we have $|x_m| < \epsilon$. Defining $\pi_m = (x_1, x_2, ..., x_m, 0, 0, ...)$ we get that $\|\pi_m(x) - x\|_\infty < \epsilon$, so $\bigcup_{n\in\mathbb{N}} S_n$ is dense. □

**1.4.45 Lemma.** *Let $(X, \tau)$ be second countable. Then*

*(1) $(X, \tau)$ is first countable.*

*(2) $(X, \tau)$ is separable.*

*Proof.* (1) Since $(X, \tau)$ is second countable, by definition, topology $\tau$ has a countable basis $\mathcal{A}$, and each element $U$ in $\tau$ is the union of elements from $\mathcal{A}$. If $x \in U$ for some $U \in \tau$, then there exists $A \in \mathcal{A}$ such that $x \in A$, where $A \subset U$. This is true for any $U$ containing $x$, so $x$ has a countable neighborhood basis.
(2) Furthermore, given a countable basis $\mathcal{A}$, for any $A \in \mathcal{A}$ we can pick some $x_A \in A$. Then $\{x_A\}_{A\in\mathcal{A}}$ is dense, because for any $U$, non-empty and open, there exists some $A \subset U$, thus $x_A \in A \subset U$. By the countability of $\mathcal{A}$, we have that $(x_A)_{A\in\mathcal{A}}$ is also countable. Thus $(X, \tau)$ is separable. □

**1.4.46 Theorem.** *Let $(X, \tau)$ be a first countable topological space. Then*

*(1) for $Y \subset X$, $a \in \overline{Y}$ if and only if there exists $(x_n)_{n\in\mathbb{N}}$ in $Y$ with $x_n \longrightarrow a$.*

*(2) a map $f : X \longrightarrow Y$ is continuous if and only if for all $x_n \longrightarrow a$, $f(x_n) \longrightarrow f(a)$.*

*Proof.* (1) Given any $a \in \overline{Y}$, by first countability there exists sequence $(U_n)_{n\in\mathbb{N}}$ of open sets (obtained from countable neighborhood system) such that for any open set $V$ with $a \in V$ there exists $n_0$ such that for all $n \geq n_0$, $U_n \subset V$. Now, since $a \in \overline{Y}$, for each $n$ we have $U_n \cap Y \neq \emptyset$. Thus, we can choose $x_n \in U_n \cap Y$ and then $x_n \longrightarrow a$.
The converse implication holds in general, for any $(X, \tau)$ topological space. Since any $(x_n)_{n\in\mathbb{N}}$ is a net, then the existence of some $(x_n)_{n\in\mathbb{N}}$ in $Y$ such that $\lim_{n\in\mathbb{N}} x_n = a$ implies, by previous Theorem, that $a \in \overline{Y}$.

(2) Now, let $f : X \longrightarrow Y$ be a continuous map. Then, for each $x_n \longrightarrow a$, $(x_n)_{n\in\mathbb{N}}$ is a convergent net, so, by continuity, $f(x_n) \longrightarrow f(a)$. Conversely, assume that for any $x_n \longrightarrow a$ we have $f(x_n) \longrightarrow f(a)$ but $f$ is not continuous. This means that there exists some filter base such that $\mathbb{B} \longrightarrow a$. But there exists open $V$ such that $f(a) \in V$ and there exists $i_0$ such that for each $i \geq i_0$, $f(B_i) \cap V = \emptyset$. Then, picking $x_i \in B_i$ gives that $f(x_i)$ does not converge to $f(a)$. □

**1.4.47 Theorem.** *A semi-metric space $(X, d)$ is first countable and is second countable if and only if is separable.*

*Proof.* Given $x \in X$, $(B_{1/n}(x))_{n\in\mathbb{N}})$ is a countable neighborhood basis. Now, assuming that $X$ is separable, let $(x_n)_{n\in\mathbb{N}}$ be a (countable) dense sequence. Then for any open open set $U \subset X$, we can write

$$U = \cup\{B_{1/n}(x_m) : (x_m \in U), B_{1/n}(x_m) \subset U\}$$

This union is indeed all of $U$, because for any $y$ in $U$, exists $\epsilon > 0$ such that $B_\epsilon(y) \subset U$ and then, by density of $(x_n)_{n\in\mathbb{N}}$, there exists $n$ so that $\frac{1}{n} < \frac{\epsilon}{3}$ and $x_m \in B_{1/n}(y)$, so $y \in B_{2/n}(x_m) \subset B_\epsilon(y)$. Thus, each open set $U \subset X$ is the union of a subset from $\mathcal{A} = \{B_{1/n}(x_m)\}_{n,m\in\mathbb{N}}$, which makes $\mathcal{A}$ a countable basis for $\tau$. The converse is true by previous Lemma: each second countable $(X, \tau)$ is separable. □

*1.4.48 Exercise.* (1) If $X$ is first-countable topological space and $A \subset X$, then $A$, equipped with the trace topology, is also first-countable.

(2) A countable product of first-countable spaces $(X_n)_{n\in\mathbb{N}}$ is also first-countable.

*Proof.* (1) Let $A \subset X$, $x \in A$ and $\mathcal{B} = \{B_1, B_2, ...\}$ a countable neighborhood basis of $X$ at $x$. Then

$$\mathcal{B}_A = \{B_1 \cap A, B_2 \cap A, ...\}$$

is a countable neighborhood basis of $A$ at x.
(2) Suppose that $x \in \prod_{n\in\mathbb{N}} X_n$, where each $X_n$ is first-countable space. Let $\mathcal{B}_n$ be a basis for $X_n$ at $x(n)$, the $n$-th coordinate of $x$. Let $J \subset \mathbb{N}$ finite and denote

$$\mathcal{B}_J = \{\prod_{n\in\mathbb{N}} B_n : B_n = X_n \text{ if } n \notin J,\ B_n \in \mathcal{B}_n \text{ if } \ n \in J\}$$

(e.g. $\mathcal{B}_{J=\{1,2\}} = \{B_1 \times B_2 \times X_3 \times X_4 \times ... : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$). Then $\mathcal{B}_J$ is countable, since it is in bijection with the set $\prod_{n\in J} \mathcal{B}_n$, a finite product of countable sets. Furthermore, the collection $\mathcal{B} = \bigcup_{\substack{J\subset\mathbb{N} \\ J finite}} \mathcal{B}_J$ is countable, because there are countably many finite subsets of $\mathbb{N}$.
Claim: $\mathcal{B}$ is a neighborhood basis for $\prod_{n\in\mathbb{N}} X_n$ at $x$.
Indeed: Take a basis element containing $x$, say $V = \prod_{n\in\mathbb{N}} V_n$, such that for the finite set $J \subset \mathbb{N}$ we have: $V_n = X_n$ if $n \notin J$ and $V_n$ open in $X_n$ for all $n$. Then, $x(n) \in V_n$, so for each $n \in J$ we can choose $B_n \in \mathcal{B}_n$ so that $x(n) \in B_n \subset V_n$. Setting $B_n = X_n$ for $n \notin J$ we get $x \in \prod_{n\in\mathbb{N}} B_n \in \mathcal{B}$. □

*1.4.49 Remark.* The statements in the last exercise hold for the more general case of second-countable spaces as well. A Proof can be found in *Munkres' Topology 2nd ed. (Theorem 30.2)*.

## 1.5 Uniqueness of limits

**1.5.50 Theorem.** *Let $(X, \tau)$ be a topological space. The Following are Equivalent:*

*(1) $X$ is Hausdorff space.*

*(2) Every convergent net $(x_i)_{i \in I}$ has a unique limit.*

*(3) Every convergent filterbase has a unique limit.*

*Proof.* $(2) \Leftrightarrow (3)$ The equivalence of nets and filterbases with respect to convergence in any topological space $(X, \tau)$ suffices to guarantee this result. Indeed, assume that every convergent filterbase has unique limit and let $(x_i)_{i \in I}$ be a net in $X$ such that $x_i \longrightarrow a$ and $x_i \longrightarrow b$, for $a \neq b$. Then $\mathbb{B} = \{B_i : i \in I\}$ where $B_i = \{x_k : i \leq k\}$ defines a filterbase with $\mathbb{B} \longrightarrow a$ and $\mathbb{B} \longrightarrow b$, which contradicts the uniqueness of the filterbases' limit. Conversely, in a topological space where each convergent net is assumed to have unique limit, if $\mathbb{B} \longrightarrow a$ and $\mathbb{B} \longrightarrow b$ was true for some filterbase $\mathbb{B}$ and some $a \neq b$, by choosing $x_B \in B$ for each $B \in \mathbb{B}$ we construct a net $(x_B)_{B \in \mathbb{B}}$ with $x_B \longrightarrow a$ and $x_B \longrightarrow b$, contradicting the uniqueness of the nets' limit.
$(1) \Leftrightarrow (2)$ Assume that $X$ is Hausdorff and $(x_i)_{i \in I} \longrightarrow a$ as well as $(x_i)_{i \in I} \longrightarrow b$. If $a \neq b$, by the Hausdorff property, we are guaranteed the existence of open sets $V_1, V_2$ such that $a \in V_1$ and $b \in V_2$ while $V_1 \cap V_2 = \emptyset$. By convergence of the net, there exists some $i_1$ such that for all $i \geq i_1$, $x_i \in V_1$. Similariry, there exists some $i_2$ such that for all $i \geq i_2$, $x_i \in V_2$. But then, if we choose some $i \geq max\{i_1, i_2\}$ we see that $x_i \in V_1 \cap V_2$ which contradicts disjointness! Hence, $a = b$.
Conversely, assume $X$ is not Hausdorff. Then we can find $a, b \in X$, with $a \neq b$, such that for any open sets $V_1, V_2$ with $a \in V_1, b \in V_2$, $V_1 \cap V_2 \neq \emptyset$ holds. Let

$$I = \{(V_1, V_2) : V_1, V_2 \text{ } open, V_1 \in U(a), V_2 \in U(b)\}$$

and let

$$(V_1, V_2) \leq (U_1, U_2) \text{ if } U_1 \subset V_1 \text{ and } U_2 \subset V_2.$$

Then, $I$ is partially ordered, directed, while choosing from each $(V_1, V_2)$ a point $x_{(V_1, V_2)} \in V_1 \cap V_2$ gives that $(x_i)_{i \in I}$ converges both to $a$ and to $b$. □

## 1.6 Compactness

**1.6.51 Definition.** A topological space $(X, \tau)$ is called:

(a) *quasi-compact*, if each open cover of $X$ has a finite subcover, i.e. if $\{U_j\}_{j \in J}$ is a family of open sets in $X$ such that $X \subset \bigcup_{j \in J} U_j$ there exists some finite $F \subset J$ such that $X \subset \bigcup_{j \in F} U_j$

(b) *compact*, if it is quasi-compact and Hausdorff.

**1.6.52 Lemma.** *(a) If $(X, \tau)$ is a topological space and $Y \subset X$ then $Y$ is quasi-compact with respect to the trace topology if and only if each open cover of $Y$ in $X$ has a finite subcover.*

*(b) If $X$ is Hausdorff and $Y \subset X$ is compact, then $Y$ is closed. If $X$ is compact and $Y = \overline{Y} \subset X$ then $Y$ is compact.*

*(c) If $X$ is normed and $Y \subset X$ is compact, then $Y$ is bounded.*

*Proof.* (a) Assume a quasi-compact $Y \subset X$, equipped with the trace topology, and an open cover $(U_j)_{j\in J}$ of $Y$, composed by open sets $U_j$ in $X$. Then, $(U_j \cap Y)_{j\in J}$ is also a cover of $Y$, composed by sets that are open in $\tau_y$. Since $Y$ is quasi-compact with respect to the trace topology, there exists a finite subcover $(U_j \cap Y)_{j\in F}$, where $|F| < \infty$. Then, $(U_j)_{j\in F}$ is a finite open cover of $Y$ in $X$.
The converse is similar. Specifically, assume $(U_i)_{i\in J}$ being an open cover of $Y$ composed by sets $U_i \in \tau_Y$. This means that each $U_i$ can be written as $U_i = V_i \cap Y$, where $V_i$ are open in $X$. Then we get

$$Y \subset \bigcup_{i\in J}(V_i \cap Y) \subset \bigcup_{i\in J} V_i$$

which implies that $(V_i)_{i\in J}$ is an open cover of $Y$ by sets $V_i$ open in $X$. By assumption, there exists a finite subcover $(V_i)_{i\in F}$ ($F$ finite) of $Y$, but then $(V_i \cap Y)_{i\in F}$ is a finite cover of $Y$ composed by sets in $(U_i)_{i\in J}$ (subcover). $\square$