Functional Analysis, Math 7320
Lecture Notes from September 22, 2016

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2.1 Separation Properties

Last time we discussed the definition of a topological vector space (TVS), and showed that every normed space is a TVS. This gives us a large number of examples, but we must remember that these are "special cases" (being *normable*), so their properties might not hold for a general TVS. In particular, one might worry about Hausdorff-ness or other separation properties.

2.1.1 Remark. Recall that any normed space is Hausdorff. This is easily shown since in the setting of a normed space we may form open balls around any point. Specifically, given $x \neq y$ in a normed space X where d is the metric induced by the norm on X, we may take $\epsilon = d(x, y) > 0$ and see that $B(x, \frac{\epsilon}{2})$ and $B(y, \frac{\epsilon}{2})$ are open neighborhoods of x and y respectively which are disjoint.

In a general TVS we cannot form open balls anymore since we do not have a metric, so the above method of proving the Hausdorff property will not directly translate into the TVS setting. Instead we use the following lemma, which gives the existence of sets that have similar properties to open balls, in that they are symmetric and can be made arbitrarily small in some sense.

2.1.2 Lemma. *If $W \in \mathcal{U}$, then there exists some $U \in \mathcal{U}$ with $U = -U$ (i.e. U is* symmetric*) and $U + U \subset W$.*

Proof. By continuity of addition at $(0, 0) \in X \times X$, there are open sets $V_0, V_0' \subset X$ with $0 \in V_0$ and $0 \in V_0'$ such that $V_0 + V_0' \subset W$. Taking $V = V_0 \cap V_0'$ gives that V is open and $V + V \subset W$. Finally, let $U = V \cap (-V)$. Then we have U is open, $U = -U$, and $U + U \subset W$. Also note $0 \in U$, so $U \in \mathcal{U}$. □

2.1.3 Remark. Given $W \in \mathcal{U}$ we may apply the above lemma to get a symmetric $U \in \mathcal{U}$ such that $U + U \subset W$. Applying the lemma now to U gives a symmetric $U' \in \mathcal{U}$ with $U' + U' \subset U$, hence $U' + U' + U' + U' \subset U + U \subset W$. For any $n \in \mathbb{N}$, clearly this process can be continued to produce a $V \in \mathcal{U}$ such that $V + \underset{(n \text{ times})}{\ldots} + V \subset W$.

To clarify notation and make our proofs more concise, we state a few facts about how neighborhoods of 0 can be used to find neighborhoods of an arbitrary set in a TVS. Coupled with the above lemma, these tell us how to find "small" neighborhoods around any point/set.

2.1.4 Remark. In any TVS X, the following are true:

(a) If $V \in \mathcal{U}$, then for any $x \in X$ the set $x + V$ is an open set containing x.

(b) If $V \in \mathcal{U}$, then for any $S \subset X$, the set $S+V$ is an open set containing S.

Proof. For (a), note that if $V \in \mathcal{U}$ then V is an open set which contains 0. By translation invariance, $x+V$ is an open set which contains x since $0 \in V$.

For (b), again we have that V is an open set which contains 0. Since $0 \in V$, we have $S \subset S+V$, and we also know that $S+V = \bigcup_{x\in S} x+V$ which is a union of open sets, hence is open. □

With these remarks, the problem of finding small enough neighborhoods of any two points in order to satisfy the Hausdorff condition can be solved by finding small enough neighborhoods of 0.

2.1.5 Theorem. *Every topological vector space is Hausdorff.*

Proof. Let X be a TVS and $x, y \in X$ with $x \neq y$. Since singletons are closed in a TVS, $\{y\}$ is closed in X, and because $x \neq y$ we have $x \in X\setminus\{y\}$ which is open. Let $W = -x + X\setminus\{y\}$, then $0 \in W$ and W is open by translation invariance of the topology (so $W \in \mathcal{U}$). By the previous lemma, there exists a $U \in \mathcal{U}$ with $U = -U$ and $U+U \subset W$. Thus $x+U+U \subset x+W = X\setminus\{y\}$. In other words:

$$\begin{aligned} & x + u_1 + u_2 \neq y && \text{for all } u_1, u_2 \in U \\ \implies & x + u_1 \neq y - u_2 && \text{for all } u_1, u_2 \in U. \end{aligned}$$

But this last line just means that $(x+U)\cap(y-U) = \emptyset$. By symmetry $y - U = y + U$, so we have $(x+U)\cap(y+U) = \emptyset$. Thus x and y can be separated by disjoint open sets, so X is Hausdorff. □

We next show a stronger separation property, which says that disjoint compact and closed sets may be separated by disjoint neighborhoods. The proof of this statement does not require using the Hausdorff property, so the Hausdorff property could have been considered a corollary of this stronger property.

2.1.6 Theorem. *Let X be a TVS, $K \subset X$ compact, and $C \subset X$ closed with $K \cap C = \emptyset$. Then there exists $V \in \mathcal{U}$ such that $(K+V)\cap(C+V) = \emptyset$.*

Proof. If $K = \emptyset$, then for any $V \in \mathcal{U}$, $K + V = \emptyset$ so we have nothing to show. So assume $K \neq \emptyset$.

For any $x \in K$, by $x \notin C$ there is an open set W_x containing x such that $W_x \cap C = \emptyset$. Letting $W'_x = -x + W_x$, we have that $W'_x \in \mathcal{U}$ and $(x + W'_x)\cap C = \emptyset$. By our remark following the previous lemma, there exists a $V_x \in \mathcal{U}$ with $V_x = -V_x$ and $V_x + V_x + V_x \subset W'_x$. Hence $(x + V_x + V_x + V_x) \cap C = \emptyset$. In other words:

$$\begin{aligned} & x + v_1 + v_2 + v_3 \neq c && \text{for all } v_1, v_2, v_3 \in V_x \text{ and } c \in C \\ \implies & x + v_1 + v_2 \neq c - v_3 && \text{for all } v_1, v_2, v_3 \in V_x \text{ and } c \in C \\ \implies & (x + V_x + V_x)\cap(C - V_x) = \emptyset \\ \implies & (x + V_x + V_x)\cap(C + V_x) = \emptyset. \end{aligned}$$

Since we may find such a $V_x \in \mathcal{U}$ for each $x \in K$, we have that $K \subset \bigcup_{x\in K}(x + V_x)$, so the collection $\{x + V_x\}_{x\in K}$ is a cover of K. Moreover, it is an open cover since each $x + V_x$ is open. By compactness of K there exists a finite subcover indexed by some $\{x_1, \dots x_n\} \subset K$, i.e. $K \subset \bigcup_{j=1}^n x + V_{x_j}$. Letting $V = \bigcap_{j=1}^n V_{x_j}$, we see that V is open and contains 0, and:

$$K + V \subset \bigcup_{j=1}^{n} (x + V_{x_j} + V) \subset \bigcup_{j=1}^{n} (x + V_{x_j} + V_{x_j}).$$

Note that $(x + V_{x_j} + V_{x_j}) \cap (C + V_{x_j}) = \emptyset$ for all $j = 1, \dots, n$. Since $C + V \subset C + V_{x_j}$ for each j, it follows that $(x + V_{x_j} + V_{x_j}) \cap (C + V) = \emptyset$ for each j, hence $(K + V) \cap (C + V) = \emptyset$. $\square$

2.1.7 Corollary. *If $\mathbb{B} \subset \mathcal{U}$ is a local base, then each $U \in \mathbb{B}$ contains $\overline{V}$ for some $V \in \mathbb{B}$.*

Proof. Let $K = \{0\}$. Given $U \in \mathbb{B}$, let $C = X \setminus U$ which is closed. Note that $0 \in U$ by definition, so $K \cap C = \emptyset$. By the above theorem there exists a $V \in \mathcal{U}$, V open, such that $V \cap (C + V) = \emptyset$ (since $K + V = V$). In other words, $V \subset X \setminus (C + V)$.

By the properties of a local base, there exists $B \in \mathbb{B}$ with $B \subset V$. Because $C + V$ is open, $X \setminus (C + V)$ is closed, hence $\overline{B} \subset \overline{V} \subset X \setminus (C + V)$. Thus $\overline{B} \cap (C + V) = \emptyset$, and since $C \subset C + V$ we have $\overline{B} \cap C = \emptyset$. In other words, $\overline{B} \subset U$. $\square$