Functional Analysis, Math 7320
Lecture Notes from September 27, 2016

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2.1 Separation Properties (continued)

From previous courses in metric spaces, we know that: Given metric spaces X, Y, a continuous map $f : X \to Y$. If a subset $K \subset X$ is compact, then $f(K)$ is compact. Upon establishing Hausdorff property of topological vector space, we can get the same result in TVS.

2.1.8 Corollary. *Given TVS X, Y, and continuous map $f : X \to Y$, $K \subset X$ compact, then $f(K)$ is compact.*

Note: $f(K)$ is compact in the trace topology $(f(K), \tau_{f(K)})$ it inherits from Y.

Proof. To get quasi-compactness, we let $(V_j)_{j\in J}$ be an arbitrary open cover of $f(K)$, which means $f(K) \subset \bigcup_{j\in J}(V_j)$, and each V_j open in Y. Hence, $(V_j \cap f(K))_{j\in J}$ forms an open cover of $f(K)$ in the trace topology $\tau_f(K)$. By continuity of f, each $U_j = f^{-1}(V_j)$ is open, and $K \subset \bigcup_{j\in J} U_j$. Since K is compact by assumption, there exists $\mathcal{F} \subset J$ such that $|\mathcal{F}| < \infty$ and $K \subset \bigcup_{j\in F} U_j$. The corresponding $(V_j \cap f(K))_{j\in F}$ forms a finite open subcover of $f(K)$ in $\tau_{f(K)}$.

Next, because Y is a TVS, it is Hausdorff (by theorem 2.1.5), so is $(f(K), \tau_{f(K)})$, the subspace topology inherited from topology of Y. We conclude that $f(K)$ is compact. □

2.2 Balancedness

2.2.9 Definition. A set K of a $\mathbb{K}$-vector space is called <u>balanced</u> if $\alpha K \in K$ for each $\alpha \in K$, with $|\alpha| \leq 1$

To reduce proofs to the special case of such neighborhoods, we need some additional insights in properties of closure and interior of sets in topological vector spaces.

Notation: $\mathcal{U}$ denotes the collection of neighborhood of 0; U° denotes the interior of U

2.2.10 Theorem. *Let X be a TVS, then we have the following results:*

(a) For $A \subset X$, $\overline{A} = \bigcap_{V\in\mathcal{U}} (A + V)$.

(b) For $A, B \subset X$, $\overline{A} + \overline{B} \subset \overline{A + B}$.

(c) If $Y \subset X$, and Y is a subspace, then $\overline{Y}$ is also a subspace of X.

(d) *If $C \in X$ is convex, then $\overline{C}$ and C° are convex.*

(e) *If $K \subset X$ is balanced, then so is $\overline{K}$. If, in addition, $0 \in K^\circ$, then K° is balanced.*

(f) *If $B \subset X$ is bounded, so is $\overline{B}$.*

Proof. (a) Given $x \in \overline{A}$. Then, $x \in \overline{A} \Leftrightarrow$ every neighborhood of x intersects with A $\Leftrightarrow$ $(x+V) \cap A \neq \emptyset$, for each $V \in \mathcal{U} \Leftrightarrow x \in A - V$. The last equivalence is because there exists an element z so that $z \in A, and z = x+y$, where $y \in V$; hence, $x = z - y \in A - V$. Now, let $U = -V$, then $U \in \mathcal{U}$ because $0 \in V$. We obtain that $x \in \overline{A}$ if and only if $x \in A + U$, for each $U \in \mathcal{U}$. Hence, $x \in \bigcup_{U \in \mathcal{U}}(A+U)$. This establishes part (a).

(b) Given $a \in \overline{A}, b \in \overline{B}$. Let $W \in \mathcal{U}(a+b)$.
Then, by continuity of addition, there exists a neighborhood $W_1 \in \mathcal{U}(a)$, and $W_2 \in \mathcal{U}(b)$ such that $W_1 + W_2 \subset W$. Apply part (a) for $a \in \overline{A}$ and $b \in \overline{B}$, there exist $x \in A \cap W_1$ and $y \in B \cap W_2$. Therefore, for each $W \in \mathcal{U}(a+b)$, $x + y \in (A+B) \cap (W_1 + W_2)$. Hence, $x + y \in (A+B) \cap W$. We conclude that $a + b \in \overline{A+B}$.

(c) Given $\alpha \neq 0$.
We recall that the map $\mathcal{M}_\alpha : X \to X$, $x \mapsto \alpha x$, is a homeomorphism, so $\alpha\overline{Y} = \overline{\alpha Y}$. In case $\alpha = 0$, then this identity is also true.
By part (b), for $\alpha, \beta \in \mathbb{K}$, $\alpha\overline{Y} + \beta\overline{Y} = \overline{\alpha Y} + \overline{\beta Y} \subset \overline{\alpha Y + \beta Y} \subset \overline{Y}$. The last relation is because Y is a subspace of X.

(d) For convexity of the closure, we can apply part (c) with $\alpha = t$, and $\beta = 1-t$. In particular, for any $t \in (0,1)$, $t\overline{C} + (1-t)\overline{C} = \overline{tC} + \overline{(1-t)C} \subset \overline{tC + (1-t)C} \subset \overline{C}$. Hence, $\overline{C}$ is convex by definition of convexity.
For interior C°, if given $t \in (0,1)$, we have $tC^\circ + (1-t)C^\circ \subset C$ (because C is convex, and $C^\circ \subset C$). However, C° is open, so $tC^\circ + (1-t)C^\circ$ is an open set. This is because $tC^\circ + (1-t)C^\circ = \bigcup_{x \in C^\circ}(tx + (1-t)C^\circ)$. The latter is an open set.
Since C° is the largest open set contained in C by definition, we conclude that $tC^\circ + (1-t)C^\circ \subset C^\circ$.

(e) Given $0 < |\alpha| \leq 1$, and $K \subset X$ be balanced, $\alpha K^\circ = (\alpha K)^\circ$ because $\mathcal{M}_\alpha$ is a homeomorphism. Since K is balanced, $\alpha K^\circ \subset \alpha K \subset K$. Moreover, αK° open implies that $\alpha K^\circ \subset K^\circ$.
Assume also that K° contains the origin, then $0K^\circ = \{0\} \subset K^\circ$. Therefore, $\alpha K^\circ \subset K^\circ$ even for $\alpha = 0$.
As for the closure, we still have $\alpha\overline{K} = \overline{\alpha K}$ because multiplication map $\mathcal{M}_\alpha$ is homeomorphism. Moreover, since K is balanced, given $0 \leq |\alpha| \leq 1$, $\alpha K \subset K$. Therefore, $\overline{\alpha K} \subset \overline{K}$. Hence, $\alpha\overline{K} \subset \overline{K}$, which implies $\overline{K}$ is balanced.

(f) Let B be bounded in X. Given $V \in \mathcal{U}$, we have $\overline{W} \subset V$ for some $W \in \mathcal{U}$ (by Cor. 2.1.7). Since B is bounded, there is $t_0 > 0$ such that for all $t > t_0, B \subset tW$. Hence, $\overline{B} \subset t\overline{W} \subset tV$. We conclude that $\overline{B}$ is bounded.

□

2.2.11 Remark. Closure $\overline{A}$ is defined to be the smallest closed sets that contains A. Part (a) provides another characterization of closure $\overline{A}$, which is the intersection of $A + V$, where V runs through the collection of neighborhood of 0. Note that, whereas $\overline{A}$ is closed, each $A + V$ is not necessarily closed

We now formulate an important consequence for the structure of topological vector spaces.

2.2.12 Theorem. *In topological vector spaces:*

(a) Each $U \in \mathcal{U}$ contains a balanced neighborhood of 0

(b) Each convex $U \in \mathcal{U}$ contains a balanced convex neighborhood of 0

Proof.

(a) Let X be a TVS. If U is a neighborhood of 0 in X, then by continuity of scalar multiplication, there is $\delta > 0$, and an open set V in X such that $B_\delta(0)V \subset U$.
Take $W = B_\delta(0)V$, then W is balanced. Moreover, $W = \bigcup_{0 \neq \lambda \in B_\delta(0)} \lambda V$, hence W is open; it also contains 0 (of X), and $W \subset V$.

(b) Suppose U is a convex neighborhood of 0 in X. Let $A = \bigcap_{|\alpha|=1} (\alpha U)$.
By part (a), U contains a balanced neighborhood W of 0. For $|\alpha| = 1$, we have that $\alpha^{-1}W = W \implies W \subset \alpha U$; hence, $W \subset A$.
Therefore, A° is an open neighborhood of 0. A is convex (because U is convex by assumption). By part (d) of Theorem 2.2.10, we have that A° is convex. What's left to show is A° is balanced. To achieve this, it suffices to show that A is balanced (balancedness of A° will follow by part (e) of Theorem 2.2.10 since αU contains 0).
Let $0 \leq r \leq 1$, $|\beta| = 1$ be given, then

$$r\beta A = \bigcap_{|\alpha|=1} r\beta(\alpha U) = \bigcap_{|\alpha|=1} r(\alpha U)$$

Since αU is a convex set that contains 0, we have that $r\alpha U \subset \alpha U$. We conclude that $r\beta A \subset A$.
□