Functional Analysis, Math 7320
Lecture Notes from September 29, 2016

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Last time, we found that in addition to having open neighborhoods within any neighborhood, we can find a *balanced* neighborhood in any neighborhood. These have the quality that they are "closed under shrinking and rotation", where the rotation is specifically the linear transformation given by a complex unit scalar. This means we can restrict our attention to balanced sets (rather than neighborhoods), which behave more like the balls in metric spaces that we are accustomed to.

In the context of this development, we take a moment to revisit the concept of boundedness in the TVS setting. Recall that a set B is bounded if for each $V \in \mathcal{U}$ there exists a positive number s such that $t \in (s, \infty)$ implies $B \subset tV$.

11.5.1 Question. If a set B has the property that for each $V \in \mathcal{U}$ there is a positive number t such that $B \subset tV$, must B be bounded? In other words, does the definition of boundedness have an equivalent formulation in "A set B is bounded if for each $V \in \mathcal{U}$ there exists a positive number t such that $B \subset tV$"?

11.5.2 Answer. Clearly, if a set is bounded, then there is at least one positive number t as in the posited definition.

Now, suppose B is a set such that every neighborhood $V \in \mathcal{U}$ has a number t_V associated with it such that $B \subset t_V$. We have shown that each $V \subset \mathcal{U}$ contains a balanced set U_V containing zero; since any such U_V is itself a neighborhood, it admits a $t_U > 0$ such that $B \subset t_U U \subset t_U V$. Let $s > t$, and set $\alpha := \frac{t}{s} < 1$. Then, noting that U balanced implies $\alpha U \subset U$,

$$B \subset t_U U = (s\alpha)U = s(\alpha U) \subset sV.$$

Hence, we may reasonably have adopted a shorter definition. As it will turn out, we will find Rudin's formulation *prescient* rather than *overburdened* as it reduces our effort in proving a number of theorems in the future.

As is often the case in mathematics, having found a "nice" object[1] that can act in the place of "fundamental" or "axiomatic" object, we turn our attention to reinterpreting some of our previous results within this new context. In particular, we can apply the argument of reducing to balanced (and convex balanced) neighborhoods to what we know about local bases.

11.5.3 Corollary (to the final theorem given in previous notes, or, equivalently, Theorem 1.14 the Second Edition of Rudin's text)**.** *Let X be a topological vector space. Then*

[1](whether simpler in some sense, easier to use, easier to find, having certain desirable qualities...)

(a) X *has a balanced local base, and*

(b) *if X is locally convex, it has a balanced convex local base.*

Of course, our closing corollary on September 22 applies here as well; that is, if $\mathcal{B} \subset \mathcal{U}$ is a *balanced* local base, then every member of $\mathcal{B}$ contains a the closure of an element in $\mathcal{B}$.

With such lovely structure in place, it is only natural to ask

11.5.4 Question. Is there an easy way to construct such "nice" local bases?

11.5.5 Answer. The following theorem will demonstrate that it is enough to have one "nice" set V having desired qualities, in which case a local base can be constructed out of scaled (and translated) copies of V.

11.5.6 Theorem. *Let V be an open neighborhood of 0 in topological vector space X.*

(a) *If $(r_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ is strictly increasing and $r_n \to \infty$, then $X = \bigcup_{n\in\mathbb{N}}(r_nV)$.*

(b) *Each compact subset $K \subset X$ is bounded.*

(c) *Given bounded set $V \subset X$ and $(\delta_n)_{n\in\mathbb{N}}$ strictly decreasing with $\delta_n \to 0$, then $(\delta_nV)_{n\in\mathbb{N}}$ is a local base of X.*

Proof. (a) Let $(r_n)_{n\in\mathbb{N}}$ be a strictly increasing number sequence with no upper bound, and fix $x \in X$. It is sufficient to show that $x \in r_nV$ for some $n \in \mathbb{N}$.

By continuity of the map $(\alpha, x) \mapsto \alpha x$, we have that the set $S := \{\alpha \in \mathbb{K} : \alpha x \in V\}$ is open in $\mathbb{K}$ (as the preimage of open set V) and contains 0. Since 0 is in the interior of S and $\frac{1}{r_n} \to 0$, it follows that for some $n_x \in \mathbb{N}$, all integers $n \geq n_x$ satisfy $\frac{1}{r_n} \in S$. Thus, $\frac{1}{r_{n_x}}x \in V$, or, equivalently, $x \in r_{n_x}V$.

(b) Let $K \subset X$ be compact, and take $W \subset V$ to be a balanced open neighborhood of 0. Then from (a), we have $K \subset \bigcup_{n\in\mathbb{N}}(nW) = X$, and note that nW is balanced for all $n \in \mathbb{N}$. Since K is compact, the open cover $\{nW\}_{n\in\mathbb{N}}$ admits an finite open subcover $\mathcal{W}$ with elements $\{nW\}_{n\in F}$ (for some finite $F \subset \mathbb{N}$); set $N := \max\{n \in F\}$. Then for $k \in F$, $\frac{k}{N} \leq 1$ and $kW = \frac{k}{N}(NW) \subset NW$. It follows that $K \subset NW \subset tNW$ for all $t > N$ (this last part by the fact that NW is balanced). Assuming X is locally bounded[2], we can take W to be bounded, and thus K is bounded.

(c) Assume $V \subset X$ is bounded and $(\delta_n)_{n\in\mathbb{N}}$ is strictly decreasing to 0, and let $U \in \mathcal{U}$. V bounded yields an $s > 0$ such that $V \subset tU$ for all $t > s$; thus, for any $n \in \mathbb{N}$ such that $s\delta_n < 1$, we have that $V \subset \frac{1}{\delta_n}U$, or, equivalently, $\delta_nV \subset U$. The existence of such an n is given by the fact $\delta_n \to 0$ implies there is a number n_s such that $n \geq n_s$ means $\delta_n < \frac{1}{s}$.

□

[2]I can find no reference nor make no connection which ensures X is locally bounded.

11.6 Linear Maps

11.6.7 Definition. Let X be a topological vector space.

(a) a linear map $f : X \to \mathbb{K}$ is called a *linear functional*[3]. We write X^* for the set of all linear functionals on X and X' for the set of all bounded linear functionals.

(b) Given a linear map $A : X \to Y$ between $\mathbb{K}$-vector spaces, the set $\mathcal{N}(A) \equiv \ker(A) \equiv A^{-1}(\{0\})$ is called the *kernel* or *null space* of A, and $\mathcal{R}(A) := A(X)$ is the *range* of A.

11.6.8 Remark. In general, as defined, a function being *continuous* means it is continuous at each point; the translation-invariance of vector topologies[4] provides yet another situation in which behavior at or near 0 determines global behavior. In particular, continuity of a linear map between topological vector spaces is equivalent to continuity at one point:

11.6.9 Theorem. *A linear map $A : X \to Y$ for topological vector spaces X, Y is continuous if and only if it is continuous at 0.*

Proof. If A is continuous, then it is, by definition, continuous at 0.

To show the reverse implication, assume A is continuous at 0 and take a neighborhood $W \in \mathcal{U}_Y(0)$. Continuity at 0 provides $V \in \mathcal{U}_X(0)$ with $A(V) \subset W$.

Fix $x \in X$ and let $x' \in X$ be such that $x' - x \in V$ (i.e., $x' \in V + x$). Then, by linearity, $A(x' - x) = Ax' - Ax \in W$. Hence, A maps $x + V$ to $Ax + W$, which means A is continuous at x. Since the only constraint on choice of x was that it was in X, A is continuous at each point of X, and thus continuous. □

We are led to a theorem which demonstrates that we can characterize continuity of linear functionals in terms of their null space:

11.6.10 Theorem. *If Λ is a nontrivial*[5] *linear functional on a topological vector space X, then the following are equivalent:*

(a) Λ is continuous.

[3]"Where does the name *functional* come from?" In the preface to his 1910 treatise "Leçons sur le calcul des variations", Jacques Hadamard expresses a bit of "hand-wringing" over his choice to use some new language in the face of "the Weierstrass tradition"[1]; in particular, we find in this text the first use of the word "functional" as defined in these notes, as the French *fonctionnel*. In chapter 37, he takes care to explain the significance of the terminology choice: "De mme qu'il nous arrivait prcdemment d'assujettir le point variable tre situ dans un certain volume, ou sur une certaine surface, etc., nous chercherons l'extremum d'une quantit qui dpend d'une ou plusieurs fonctions arbitraires en assujettissant celles-ci un certain nombre de conditions, ou, comme nous dirons souvent, se trouver dans un certain champ fonctionnel. Cette dernire locution (manifestement inspire par l'analogie avec ce qui se passe pour les extrema ordinaires) ne devra d'ailleurs tre considre que comme un synonyme de la premire: le champ fonctionnel est, par dfinition, l'ensemble des fonctions (ou des systmes de fonctions) qui satisfont aux conditions donnes."

Not surprisingly, in his essay *The Language Crisis*[2], Hadamard takes the stance that mathematicians, in particular, should work very hard to avoid multiple meanings for the same technical word (lamenting that "conjugate" has two meanings even to the algabraist), as well as many words for the same concept ("integral" used as the *solution* of an integral, for example). I suppose it is a testament to his dire concern for consistency and clarity that we still use his word today, even in the absence of understaning its source.

[4]Recall that T_a and M_λ are homeomorphisms of a topological vector space X onto itself.

[5]That is, there exists an $x \in X$ such that $\Lambda x \neq 0$.

(b) $\mathcal{N}(\Lambda)$ is closed.

(c) $\mathcal{N}(\Lambda)$ is not dense in X.

(d) Λ is bounded on a neighborhood of 0.

Proof. We shall show that (a)⇒(b)⇒(c)⇒(d)⇒(a).

(a)⇒(b): Assume Λ is continuous. Then $\mathcal{N}(\Lambda) = \Lambda^{-1}(\{0\})$ is the continuous preimage of a closed set, and is therefore closed. Thus (a) implies (b).

(b)⇒(c): Assume $\mathcal{N}(\Lambda)$ is closed. By the nontriviality assumption, $\mathcal{N}(\Lambda) = \overline{\mathcal{N}(\Lambda)} \neq X$, so $\mathcal{N}(\Lambda)$ is not dense in X, and (b) implies (c).

(c)⇒(d): Assume $\mathcal{N}(\Lambda)$ is not dense in X, and so $X \backslash \mathcal{N}(\Lambda)$ has a nonempty interior. We take $x \in (X \backslash \mathcal{N}(\Lambda))^o$, and select a balanced neighborhood V of 0 such that $(x+V) \cap \mathcal{N}(\Lambda) = \emptyset$. By linearity of Λ, $\Lambda(V) \subset \mathbb{K}$ is balanced, which implies that either $\Lambda(V) = \mathbb{K}$ or $\Lambda(V)$ is bounded.

Suppose $\Lambda(V) = \mathbb{K}$. Then $\Lambda^{-1}(\{\Lambda x\}) \cap V$ is nonempty and there exists $y \in V$ such that $\Lambda y = -\Lambda x$. By linearity of Λ, $\Lambda(x+y) = 0$, from which we discern $x + y \in \mathcal{N}(\Lambda)$. This contradicts our selection of V, and thus our supposition is false. Thus $\Lambda(V)$ is bounded and (c) implies (d).

(d)⇒(a): By Theorem 11.6.9, it suffices to show that (d) implies that Λ is continuous at 0. We begin by assuming (d) holds; that is, there is positive number M and a neighborhood V of 0 such that $|\Lambda x| < M$ for all $x \in V$. Let $\epsilon > 0$ and set $W := \frac{\epsilon}{M} V$. Then the linearity of Λ yields that $x \in W$ implies $|\Lambda x| < \frac{\epsilon}{M} M = \epsilon$.

□

References

[1] Hadamard, Jacques, *Leçons sur le calcul des variations*, Paris, 1910.

[2] Hadamard, Jacques, "The Language Crisis", *Erkenntnis*, Jan 01, Vol 7, 1937.