Functional Analysis, Math 7320
Lecture Notes from October 6, 2016

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Last time:

1. Finite dimensional subspaces and linear maps
2. Closedness of finite dimensional subspaces
3. Finite dimensionality of spaces with locally compact bases

2.7.1 Problem. Suppose X and Y are topological vector spaces with $n = \dim(Y) < \infty$ and $A\colon X \to Y$ is linear and onto. Show that A is an open map.

Proof. Y is homeomorphic to $\mathbb{K}^n$, which means that we may assume without loss of generality that $Y = \mathbb{K}^n$. Now choose an arbitrary open set $U \subset X$. To show that $A(U)$ is open, we will show that for any point $Ax \in A(U)$ there exists $N \in \mathcal{O}(Ax)$ such that $N \subset A(U)$. This will imply that the arbitrary point Ax is an interior point of $A(U)$, and hence, that $A(U)$ is open. The idea of the proof is to restrict A so that it becomes continuous.

Since A is onto, we can choose elements $\{x_j\}_{j=1}^n \subset X$ such that $Ax_j = e_j$ for $1 \leq j \leq n$. Here $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{K}^n$. Now define a map $B\colon \mathbb{K}^n \to X$ by $Ba = \sum_{i=1}^n a_j x_j$ where $\{a_j\}_{j=1}^n$ are the coordinates of a with respect to the standard basis. From prior lessons we know that B is linear and continuous so that B^{-1} is an open map.

Observe that $B^{-1}(U) \subset A(U)$. This follows because if $a = \langle a_1, \ldots a_n\rangle \in B^{-1}(U)$ then $Ba = \sum_{j=1}^n a_j x_j$. Therefore $A(\sum_{j=1}^n a_j x_j) = \sum_{j=1}^n a_j e_j = a$ so that $a \in A(U)$.

Now assume that $0 \in U$. It follows that $B^{-1}(U) \in \mathcal{O}(0)$ so that $A(U)$ contains an open neighborhood of 0. If $0 \notin U$, then choose any vector $y_0 \in A(U)$ and find x_0 so that $Ax_0 = y_0$ by the surjectivity of A. If $\tilde{U} = U - x_0$ then $\tilde{U} \in \mathcal{O}(x)$. By what we have just shown there exists $\tilde{V} \in \mathcal{O}(0)$ such that $\tilde{V} \subset A(\tilde{U})$. But then $V = \tilde{V} + Ax_0 \in \mathcal{O}(Ax_0)$ and $V \subset A(U)$. □

2.7.2 Lemma. *Suppose X is a vector space, $\{X_i\}$ is a family of topological vector spaces, and that $\{f_i\colon X \to X_i\}$ is a family of maps which separates points. Then X is a topological vector space with the initial topology.*

Proof. Each map f_i is continuous with respect to the initial topology on X by the definition of the initial topology. And for any $x \in X$ and i we know that $\{f_i(x)\}$ is a closed set since X_i is T_1. Therefore, since f pulls closed sets back to closed sets, we have that $\{x\} = \bigcap_i f_i^{-1}(\{f_i(x)\})$ is an intersection of closed sets, and hence, is closed. The set equality follows because the family

$\{f_i\}$ separates points; hence, if $y \in \bigcap_i f_i^{-1}(\{f_i(x)\})$, then for every i we have $f_i(y) = f_i(x)$ which forces $y = x$.

Recall that the function $\eta\colon X \to \prod_i X_i$ given by $\eta(x) = (f_i(x))_i$ is continuous and linear. This is because the mapping $(x, y) \mapsto f_i(x+y) = f(x)+f(y)$ is a composition of the continuous operation of addition in X_i and the map $x \mapsto f_i(x)$, and so is continuous with respect to the product topology. Similarly the mapping $(\alpha, x) \mapsto \alpha f_i(x) = f_i(\alpha x)$ is continuous.

Now we can identify X with $\eta(X) \subset \prod_i X_i$. Thus X inherits the relative topology from the product topology on $\prod_i X_i$ and so becomes a topological vector space by what we have shown. □

If we apply this result to the case in which $X = \prod_i X_i$ and $f_i = \pi_i$ is projection onto the i-th coordinate, we get the following.

2.7.3 Corollary. *The Cartesian product of a family of topological vector spaces is itself a topological vector space. Furthermore, with f_i, X, and X_i as in* **??**, *the map $\eta\colon x \mapsto (f_i(x))_i$ gives an embedding of X into $\prod_i X_i$.*

2.7.4 Lemma. *(a) Given a semi-norm p on a vector space X, the set $N_p = \{x\colon p(x) = 0\}$ is a subspace of X and X/N_p is a normed space with norm $\|x + N_p\| = p(x)$.*

(b) If $f\colon X \to Y$ is linear map from a vector space X to a normed linear space Y, then $p_f(x) = \|f(x)\|$ defines a semi-norm on X, and every semi-norm on X is obtained in this way via a function $f\colon X \to Y$ and a normed linear space Y.

Proof. For part (a) we need only show positive-definiteness. So suppose $\|x + N_p\| = 0$. then $p(x) = 0$ so that $x \in N_p$ and hence $x \sim 0$. Thus $x + N_p$ is the 0 element in X/N_p.

For part (b), the verification that p_f is a semi-norm on X is trivial and follows from the properties of the norm on Y. For the final claim, choose a semi-norm p on X and define $Y = X/N_p$ with the quotient norm. Then the canonical quotient map $q\colon x \mapsto x + N_p$ is the desired function f which makes the equality $p_f(x) = p(x)$ hold. □

2.7.5 Definition. A family of semi-norms $\{p_i\}$ on a vector space X is said to *separate points* if $\bigcap_i N_{p_i} = \{x\colon p_i(x) = 0 \text{ for all } i\} = \{0\}$.

2.7.6 Theorem. *If $\{p_i\}_{i\in I}$ is a family of semi-norms on a vector space X which separates points, for $\epsilon \in \mathbb{R}$ and $i \in I$ define a set $V(p_i, \epsilon) = \{x\colon p_i(x) < \epsilon\}$. Then*

$$\mathcal{B} = \{V(p_{i_1}, \epsilon) \cap \cdots \cap V(p_{i_n}, \epsilon)\colon \epsilon > 0,\ \{i_1, \ldots, i_n\} \subset I\}$$

is a local base of the vector space topology induced by $\{p_i\}$.

Proof. For $i \in I$ let $q_i\colon X \to X/N_{p_i}$ be the canonical quotient maps. By **??**, X/N_{p_i} is a normed linear space for each $i \in I$ with the quotient norm defined there. The quotient maps $\{q_i\}$ give rise to an initial topology τ on $\mathbb{X}$. Now any sequence of open balls centered at 0 whose radii decrease to 0 defines a local base in X/N_{p_i} for any $i \in I$. Recall that τ is generated by arbitrary unions of finite intersections of open sets of the form $q_i^{-1}(U_i)$ where U_i is open in X/N_{p_i}. So if $x \in X$ and $U \in \mathcal{O}(x)$ then U contains a basis element for τ, namely, a finite intersection of sets $V(p_i, \epsilon)$. So for every $U \in \mathcal{X}$ there exists $B \in \mathcal{B}$ with $B \subset U$. □

2.7.7 Corollary. *The collection $\{p_i\}$ in **??** define a locally convex topology on X.*

Proof. $V(p_i, \epsilon)$ is the inverse image of a ball under a linear map and so is convex. Thus any finite intersection of sets of this form is convex. □

2.7.8 Definition. Suppose X is a vector space.

(a) For $A \subset X$ define the Minkowski functional $\mu_A \colon X \to [0, \infty]$ by $\mu_A = \inf\{t > 0 \colon x \in tA\}$.

(b) $A \subset X$ said to be absorbing if $X = \bigcup_{t>0} tA$.

Note that a set A is absorbing if and only if μ_A never takes the value $+\infty$. Also, every absorbing set contains 0.