Functional Analysis, Math 7320
Lecture Notes from October 13, 2016

taken by Robert Mendez

11.5.1 Theorem. *Let $\mathcal{P}$ be a family of seminorms on vector space X that separate points. For $p \in \mathcal{P}$, let $V(p, \frac{1}{n}) := \{x : p(x) < \frac{1}{n}\}$ and let $\mathbb{B}$ be the collection*

$$\mathbb{B} := \left\{ \bigcap_{i=1}^{n} V\left(p_i, \frac{1}{n}\right) : n \in \mathbb{N}, i \in I \subset \mathbb{N}, |I| = n \right\}.$$

Then

(a) $\mathbb{B}$ is a convex balanced local base for a topology τ on X which makes X a locally convex topological vector space,

(b) each p is continuous, and

(c) each $E \subset X$ is bounded if and only if each p is bounded on E.

Proof. Let $\mathcal{P}, X, V, \mathbb{B}$ be as in the hypothesis, and take τ to be the topology on X generated by subbase $X + \mathbb{B}$, the collection of sets of the form $x + \bigcap_1^n V(p_i, \frac{1}{n})$.

Since each set $V(p, \frac{1}{n})$ is the inverse image of a (convex) ball by a linear map, we have that $\mathbb{B}$ is comprised by convex balanced sets, and thus bB is a convex balanced base for τ.

To show that τ endows X with the properties of a topological vector space, we examine τ's construction. Note first that by definition, each $V(p, \frac{1}{n})$ is open. We further observe that $X + \mathbb{B}$ generates a base $\mathcal{B}$ for τ by finite intersections, so that $A \in \mathcal{B}$ implies that A is of the form $\bigcap_{i=1}^m (x_i + B_i)$, $B_i \in \mathbb{B}$; since the elements $B_i \in \mathbb{B}$ are themselves finite intersections, we can express $A \in \mathcal{B}$ as

$$A = \bigcap_{i=1}^{m} \left[x_i + V\left(p_i, \frac{1}{n_i}\right) \right]$$

with each triple $(x_i, p_i, n_i) \in X \times \mathcal{P} \times \mathbb{N}$.

Now, we demonstrate that points are closed. If $x \in X \backslash \{0\}$, then the assumption that $\mathcal{P}$ separates points provides the existence of $p \in \mathcal{P}$ such that $p(x) > 0$. Let n be an integer greater than $1/p(x)$, so that $p(x) > 1/n$. Then $x \notin V(p, \frac{1}{n})$ and $0 \in V(p, \frac{1}{n})$ together give that $0 \notin x - V(p, \frac{1}{n}) \in \tau$. It follows that $x \notin \overline{\{0\}}$ for all nonzero $x \in X$, and we conclude that $\overline{\{0\}} = \{0\}$. By translational invariance, $\{y\}$ is closed for each $y \in X$.

Next, we note that $U \in \mathcal{U}$ implies $0 \in V(p_1, \frac{1}{n_1}) \cap \cdots \cap V(p_m, \frac{1}{n_m}) \subset U$ for some $\{p_i\}_1^m, \{n_i\}_1^m$. To see this, we consider $W \subset U$ such that W is an open set containing zero; We may take W to be the finite intersection of sets $V_i := y + V(p_i, \frac{1}{n_i})$, each containing 0. It follows that $-y \in V(p_i, \frac{1}{n_i})$. Since the $V(p_i, \frac{1}{n_i})$ are open, we can choose $\epsilon \in (0, 1/n_i)$, giving $-y + V(p_i, \frac{1}{\epsilon}) \subset V(p_i, \frac{1}{n_i})$. Thus, $0 \in V(p_i, \frac{1}{\epsilon}) \subset y + V(p_i, \frac{1}{n_i})$;

To show continuity of addition, let $V := V(p_1, \frac{1}{2n_1}) \cap \cdots \cap V(p_m, \frac{1}{2n_m})$, so that $V + V \subset U$ (by the subadditivity of the p_i's). It follows that the vector sum is continuous at 0, and therefore continuous everywhere.

Next, keeping U, V as above, then if $x \in X, \alpha \in \mathbb{K}$,we have $x \in sV$ for some $s > 0$.

Set $t := \frac{s}{1+|\alpha|s}$, and let $y := x + tV$. We take $\beta \in \mathbb{K}$ such that $|\beta - \alpha| < 1/s$, notably giving $|\beta| < |\alpha| + 1/s$. Then

$$\begin{aligned} \beta y - \alpha x &= \beta(y - x) + (\beta - \alpha)x \quad \text{(by def'n, } y - x \in tV) \\ &\in |\beta| tV + |\beta - \alpha| sV \\ &\subset V + V \\ &\subset U. \end{aligned}$$

Thus, $(\alpha, x) \mapsto \alpha x$ is continuous. We turn our attention to (b). Let $p \in \mathcal{P}$ and note that, by definition, $V(p, 1)$ is a neighborhood of 0. Scaling by $\alpha > 0$, we observe that by homogeneity of the seminorm,

$$\alpha V(p, 1) = \{x \in X : p(x) < \alpha\} \quad (1)$$
$$= p^{-1}\left(B_\alpha(0)\right). \quad (2)$$

Now, given $\epsilon > 0$, $x \in X$, and $r := p(x)$ we have that for $q \in (0, \epsilon)$, $p(x + qV(p, 1)) \subset r + B_\epsilon(0)$. □

11.5.2 Corollary. *If (X, τ) is a topological vector space with balanced local base $\mathbb{B}$ of open sets, then the seminorms $\mathcal{P} = \{\mu_r\}_{r \in \mathbb{B}}$ induce a topology which is identical to that of X.*

Proof. Let τ_1 be the topology induced by $\mathcal{P}$. by continuity of the seminorms, we have that $V(p, \frac{1}{n}) \in \tau$ for all $p \in \mathcal{P}$, so $\tau_1 \subset \tau$.

On the other hand, if $W \in \mathbb{B}$, then $p = \mu_W$ and

$$\begin{aligned} V(p, 1) &= \{x : \mu_W(x) < 1\} \\ &= W. \end{aligned}$$

Thus, $W \in \mathbb{B}$ implies $W \in \tau_1$, and so $\tau \subset \tau_1$ and the claim is proven. □

11.5.3 Remark. It is natural to wonder why such constructs should be necessary. The following example describes an application which supports what are known as "Dirichlet problems", of interest in PDEs and harmonics.

11.5.4 Example. Consider a set Ω, open in $\mathbb{R}^d$; by normality of $\mathbb{R}^d$, there exists a nested sequence of compact sets $K_1 \subset K_2 \subset \cdots$ and a nested sequence of open sets $U_1 \subset U_2 \subset \cdots$ such that $K_j \subset U_j \subset K_{j+1}$ for $j \in \mathbb{N}$ with the property that $\bigcup_1^\infty K_j = \Omega$. Let $C(\Omega)$ be the space of all continuous functions on Ω. Since $f \in C(\Omega)$ can diverge, the "sup norm" $\|\cdot\|_\infty$ cannot be defined on all of $C(\Omega)$.

Instead, we let $p_j(f) = \sup\{|f(x)| : x \in K_j\}$; then each p_j is a seminorm and $p_1 \leq p_2 \leq \cdots$. We get $V_n := \{f \in C(\Omega) : p_n(f) < 1/n\}$ forms a local base. By the topology induced by the seminorms, $C(\Omega)$ becomes a locally convex topological vector space. However–and this is where part (c) in Theorem 11.5.1 becomes important–each $U \in \mathcal{U}$ contains $V := \{f \in C(\Omega) : p_N(f) < 1/m\}$, $N, m \in \mathbb{N}$, but then we can choose for any $L > 0, N' > N$ a function $f \in V$ with $\max |f|_{K_{N'}}| > L$, so $p_{N'}$ is not bounded on V.

We see, then that there is no chance of defining a norm on $C(\Omega)$ with the structure of a topological vector space. This sentiment is made rigorous in the statement of Corollary 11.5.7.

11.5.5 Theorem. *A topological vector space is normable if and only if* 0 *has a convex bounded neighborhood.*

Proof. If τ comes from a norm, then $B_1(0) = \{x \in X : \|x\| < 1\}$ gives a desired convex bounded neighborhood; this follows from the fact that if V is an open neighborhood of 0, there exists $\epsilon > 0$ such that $\epsilon B_1(0) \subset V$, so $B_1(0) \subset \frac{1}{\epsilon} V$.

Conversely, let V be a convex bounded neighborhood of 0. We know there exists a convex, balanced open neighborhood U of 0 in V. Let, for $x \in X$, $\|x\| = \mu_U(x)$; by virtue of U being balanced and convex, μ_U is a seminorm. Since U is bounded, $(rU)_{r>0}$ is a local base. For nonzero x, there exists $q > 0$ such that $x \in qU$, and so $\|x\| \geq r$. Since μ_U is a seminorm and $\|x\| > 0$ for all nonzero $x \in X$, $\|\cdot\|$ is a norm. □

11.5.6 Remark. We have, then, that $\{B_r(0)\}_{r>0}$ giving a local base actually characterizes the topology, so the norm topology is identical to the original topology on X.

11.5.7 Corollary. *Since* $C(\Omega)$ *does not have a bounded neighborhood of* 0*, it is not normable.*