Functional Analysis, Math 7320
Lecture Notes from October 25, 2016

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Warm up

Let $\mathbb{Z}_2^\infty := \bigcup_{n=1}^\infty \left(\mathbb{Z}_2^n \times \{0\}^\infty\right)$ denote the family of sequences of 0's and 1's having only finitely many nonzero terms.[1] Further, let D be the set of values in $[0,1)$ that can be represented as the ratio of a whole number and a power of 2, known as the *dyadic rationals* and explicitly defined

$$D := \left\{ \sum_{j=1}^n \frac{c_j}{2^j} : c_j \in \{0,1\}, n \in \mathbb{N} \right\}.$$

We note that we may see any point $x = (x_j)_1^\infty$ in $\mathbb{Z}_2^\infty$ as a point in D simply by a replacing the c_j's in the dyadic expression for the x_j's, with $c_j = x_j$ for all $j \in \mathbb{N}$; that is, let

$$\begin{aligned} \beta : \mathbb{Z}_2^\infty &\to D \\ (x_j)_1^\infty &\mapsto \sum_{j=1}^n \frac{x_j}{2^j} \end{aligned}$$

In this way, we may see the points of $\mathbb{Z}_2^\infty$ as the binary "decimal" representation–perhaps "*bicimal*"?–of the dyadic rationals. The map β is not a homomorphism, for addition in $\mathbb{Z}_2^\infty$ (as a vector space over the field $\mathbb{Z}_2$) occurs only component-wise, whereas in D and in the bicimals, addition may involve "carrying." Consider the points $x = (1,0,0,1,0,0,...)$ and $y = (0,1,0,1,0,0,...)$ in $\mathbb{Z}_2^\infty$, then their sum is

$$\begin{array}{rr} x & (1,0,0,1,0,0,...) \\ \underline{+y} & \underline{+(0,1,0,1,0,0,...)} \\ x+y & (1,1,0,0,0,0,...) \end{array}$$

while

$$\begin{array}{rr} \beta(x) & .1001_2 \\ \underline{+\beta(y)} & \underline{+.0101_2} \\ \beta(x+y) & .1110_2 \end{array}$$

[1] This notation is borrowed from the standard distinction between $\prod_1^\infty \mathbb{R} =: \mathbb{R}^\omega \equiv \mathbb{R}^{\aleph_0} \equiv \mathbb{R}^{|\mathbb{N}|}$ and $\mathbb{R}^\infty \subset \mathbb{R}^\omega$, used in [**?**], for example.

In proving today's primary theorem, we may think of the points in $\mathbb{Z}_2^\infty$ as being endowed with the addition given by $x \oplus_2 y$ defined $\beta^{-1}\big(\beta(x)+\beta(y)\big)$.

Warm up 2

Last time, we concluded a characterization of locally convex topological vector spaces in terms of an inducing family of seminorms. As we begin the final section of this chapter of study, we recall that each topological vector space X is *regular*–that is, given a point $x \in X$ and a neighborhood U of x, there is a neighborhood V of x whose closure is in the interior of U[2]. We demonstrated this in our discussion on separation properties, using the continuity of addition to give the existence of neighborhoods which sum within specific neighborhoods. Explicitly, choosing $x = 0$ without loss of generality and given $V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W + W \subset V$. In fact, repeated applications provide that, for any $n \in \mathbb{N}$, there exists $W' \in \mathcal{U}$ such that $\underbrace{W' + W' + \cdots + W'}_{n \text{ summands}} \subset V$.

In particular, we will consider a sequence $\{V_n\}$ of open, balanced neighborhoods of 0 such that each V_n subsumes the sum of four copies of V_{n+1}. This warm up prepares us for the proof of our first theorem as we begin our discussion on metrization of topological vector spaces.

3 Metrization

Recall that a topological space (X, τ) is said to be metrizable if there exists a metric on X which induces its topology. In other words, the local balls given by a metric d on X provide a basis for τ. In such a case, the open balls $\big(B_{\frac{1}{n}}(0)\big)_{n\in\mathbb{N}}$ give a local base for X; thus, the existence of a countable local base is a necessary condition for metrizability in a topological vector space.

As it turns out, this condition is also sufficient. Rudin makes use of the existence of the "self-summing subsets" in any TVS neighborhood mentioned above to prove the following theorem.

3.0.1 Theorem. *Let X be a topological vector space with a countable local base. Then there exists a metric d such that*

(a) every open set in τ is the union of open balls with respect to d;

(b) the open balls $\{B_r(0)\}_{r>0}$ are balanced;

(c) d is (translation) invariant, so that for $x, y, z \in X$, $d(x+z, y+z) = d(x, y)$;

(d) if X is locally convex, then d can be chosen so that the open balls $\{B_r(0)\}_{r>0}$ are convex.

3.0.2 Remark. As alluded to in the warm up, we shall make use of a family of open, balanced sets with some proscribed properties. Before we begin our proof proper, the following lemma introduces a useful property of such families.

[2]A regular topological space is equivalently characterized as having the property that any closed set and point outside of it can be separated by disjoint open sets; this is the essence of Theorem 2.1.6 in the 22 September 2016 notes.

3.0.3 Lemma. *Let X be a topological vector space. Define $V_0 := X$ and suppose $\{V_n\}_{n=0}^{\infty}$ is a family of open, balanced neighborhoods of 0 with the property that*

$$V_n + V_n + V_n + V_n \subset V_{n-1} \quad \textit{for all } n \in \mathbb{N}.$$

Then, for nonnegative integer n and any finite increasing sequence of natural numbers $j_1, j_2, \dots, j_N$,

$$\begin{aligned} V_{n+j_1} + V_{n+j_2} + \cdots + V_{n+j_N} \subset \\ V_{n+j_1} + V_{n+j_1} + V_{n+j_2} + V_{n+j_2} + \cdots + V_{n+j_N} + V_{n+j_N} \subset V_n. \end{aligned}$$

Proof. Let $\{V_n\}$ be as in the hypothesis. The first inclusion is immediate, and it is clear that $V_{n+j} \subset V_n$ for all $j \in \mathbb{N}$; we offer an inductive proof, beginning with the case where $N = 2$.

($N = 2$) Let n be a nonnegative integer and $j_1 < j_2$ be natural numbers. Then

$$\begin{aligned} V_{n+j_1} + V_{n+j_1} + V_{n+j_2} + V_{n+j_2} &\subset V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} && \text{(since } j_1 \geq 1) \\ &\subset V_n. && \text{(by hypothesis)} \end{aligned}$$

(Induction hypothesis) Assume that for $n \in \mathbb{N}$ and any increasing indexed set $\{j_i\}_1^N \subset \mathbb{N}$ of order at most K we have that

$$V_{n+j_1} + V_{n+j_1} + V_{n+j_2} + V_{n+j_2} + \cdots + V_{n+j_N} + V_{n+j_N} \subset V_n.$$

Let n be a nonnegative integer and $\{j_i\}_1^{K+1} \subset \mathbb{N}$ be increasing. Then the set $\{j_i\}_2^{K+1}$ is of order K and

$$\begin{aligned} V_{n+j_1} + V_{n+j_1} + \overbrace{V_{n+j_2} + V_{n+j_2} + \cdots + V_{n+j_{(K+1)}} + V_{n+j_{(K+1)}}}^{2K \text{ terms}} &\subset V_{n+j_1} + V_{n+j_1} + V_{n+j_2-1} \\ &\subset V_{n+j_1} + V_{n+j_1} + V_{n+j_1} \\ &\subset V_n. \end{aligned}$$

□

We now proceed with the proof of the main theorem.

Proof. We shall induce a partition on X which defines a distance from each point to 0, from which the distance between arbitrary points will be given by translation. To develop our partition, we start with a countable local basis at 0 that "shrinks fast enough" to exhibit some useful qualities.

Take $V_1 \in \mathcal{U}$ to be any balanced, open neighborhood of 0, and set $V_0 := X$. For $n \in \mathbb{N}$, let $V_{n+1} \in \mathcal{U}$ be balanced and open with the property that

$$V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subset V_n,$$

for which we have already shown existence. The family $\{V_n\}_{n\in\mathbb{N}}$ is a balanced local base for X, and if X is locally convex, each V_n may be taken to be convex.

Define $\mathcal{N} \in \mathcal{P}(\mathbb{N})$ to be the the family of finite subsets of $\mathbb{N}$, $\mathcal{N} = \{J \in \mathbb{N} : |J| < \infty\}$, and let $P := V_0 \cup \{A \in \mathcal{U} : A = V_{j_1} + V_{j_2} + \cdots + V_{j_N}, j_i \in J \subset \mathcal{N}\}$, the family of finite sums of elements in our local base, taken without summand repetition. Associate with each $A \in P$

the index set $J_A \subset \mathbb{N}$, so that we may write $A = \sum_{j \in J_A} V_j$. By construction of $\{V_n\}$ and the preceding lemma, we note that for $A \in P$ with increasing indices $j_i \in J_A$,

$$\begin{aligned} A &= V_{j_1} + V_{j_2} + \cdots + V_{j_N} \\ &\subset V_{j_1 - 1}, \end{aligned}$$

Additionally, note that the elements of P are totally ordered under set inclusion. That is, for distinct $A, B \in P$, either $A \subsetneq B$ or $B \subsetneq A$. Taking $(j_i)_{i=1}^{N_1} = J_A$ and $(k_i)_{i=1}^{N_2} = J_B$ to be increasing, we compare elements j_i and k_i and identify the least index η such that $j_\eta \neq k_\eta$. Since $\sum_{i=1}^{\eta-1} V_{j_i} = \sum_{i=1}^{\eta-1} V_{k_i}$, we have that

$$\begin{array}{ccccc} \displaystyle\sum_{i=1}^{\eta-1} V_{j_i} + V_{j_\eta} & \subset & A & \subsetneq & \displaystyle\sum_{i=1}^{\eta-1} V_{j_i} + V_{j_\eta - 1} \\ & & \text{and} & & \\ \displaystyle\sum_{i=1}^{\eta-1} V_{j_i} + V_{k_\eta} & \subset & B & \subsetneq & \displaystyle\sum_{i=1}^{\eta-1} V_{j_i} + V_{k_\eta - 1}. \end{array}$$

Without loss of generality, assume $j_\eta > k_\eta$. Then $V_{j_\eta} \subsetneq V_{k_\eta}$ and $V_{j_\eta - 1} \subset V_{k_\eta}$, and we have

$$A \quad \subsetneq \quad \sum_{i=1}^{\eta-1} V_{j_i} + V_{j_\eta - 1} \quad \subset \quad \sum_{i=1}^{\eta-1} V_{j_i} + V_{k_\eta} \quad \subset \quad B,$$

giving the strict inclusion of the claim.

This total ordering is key in producing the partition we set as our initial goal, but first we give our sets $A \in P$ an indexing that more clearly expresses the ordering.

Let $\mathbb{Z}_2^\infty := \bigcup_{n=1}^\infty \left(\mathbb{Z}_2^n \times \{0\}^\infty \right)$, as in the warm up. Then the function

$$\begin{aligned} \phi : P &\to \mathbb{Z}_2^\infty \\ A &\mapsto \phi(A) \end{aligned}$$

defined by the coordinate functions

$$\phi_j(A) = \begin{cases} 1 & \text{if } j \in J_A, \\ 0 & \text{otherwise} \end{cases}$$

gives a natural bijection between $P \backslash V_0$ and the dyadic rationals, which we extend to all of P via $\rho := \beta \circ \phi$:

$$\begin{aligned} \rho : P &\to D \cup \{1\} \\ A &\mapsto \begin{cases} 1 & \text{if } A = V_0, \\ \sum_{j=1}^n \frac{1}{2^j} \phi_j(A) & \text{otherwise.} \end{cases} \end{aligned}$$

We see that ρ is monotonic, giving $\rho(A) < \rho(B)$ if and only if $A \subset B$ for all $A, B \in P$, and thus ρ preserves the total ordering on P. For any $r \in D \cup \{1\}$, we may write A_r to indicate the set $\rho^{-1}(r)$.

This provides our partition, given by the quotient map

$$\begin{aligned}\pi : X &\to [0,1]\\ x &\mapsto \inf_{A \in P}\{\rho(A) : x \in A\},\end{aligned}$$

from which we define our metric

$$d(x,y) := \pi(x-y).$$

Before we prove that d satisfies our requirements, we state and prove the following claim: For $r, s \in D$, $A_r + A_s \subset A_{r+s}$.

We consider $\phi(A_r), \phi(A_s) \in \mathbb{Z}_2^\infty$, and let $k \in \mathbb{N}$ be the least index such that $\phi_k(A_r) = \phi_k(A_s) = 1$. The lemma ensures that

$$\begin{aligned}A_r + A_s &= \sum_{j=1}^{\infty} \phi_j(A_r)V_j + \sum_{j=1}^{\infty} \phi_j(A_s)V_j\\ &= \sum_{j=1}^{k-1} \big(\phi_j(A_r) + \phi_j(A_s)\big)V_j + \underbrace{V_k + V_k}_{\substack{\text{since } \phi_k(A_r)\\ = \phi_k(A_s) = 1}} + \sum_{j=k+1}^{\infty} \underbrace{\big(\phi_j(A_r) + \phi_j(A_s)\big)V_j}_{\substack{\text{each } V_j \text{ appears as a}\\ \text{summand at most twice}}}\\ &\subset \sum_{j=1}^{k-1} \big(\phi_j(A_r) + \phi_j(A_s)\big)V_j + V_{k-1};\end{aligned}$$

Note that adding r and s in binary notation means that $\phi_k(A_r) = \phi_k(A_s) = 1$ forces a "carry"; this gives rise to two cases.

Case 1 If $\phi_{k-1}(A_r) = \phi_{k-1}(A_s) = 0$, then that carried value sets $\phi_{k-1}(A_{r+s}) = 1$ and $\phi_j(A_{r+s}) = \phi_j(A_r) + \phi_j(A_s)$ for $j \in [\![k-1]\!]$. If If $\phi_{k-1}(A_r) = \phi_{k-1}(A_s) = 0$, it follows that $\phi_{k-1}(A_{r+s}) = 1$ and

$$\begin{aligned}\sum_{j=1}^{k-1} \big(\phi_j(A_r) + \phi_j(A_s)\big)V_j + V_{k-1} &= \sum_{j=1}^{k-1} \phi_j(A_{r+s})\\ &\subset \sum_{j=1}^{\infty} \phi_j(A_{r+s})\\ &= A_{r+s}.\end{aligned}$$

Case 2 On the other hand, if $\phi_{k-1}(A_r) \neq \phi_{k-1}(A_s)$, or, equivalently, $\phi_{k-1}(A_r) + \phi_{k-1}(A_s) = 1$, then this induces another carry; the number of carries depends on the number of consecutive 1's in the binary notation.

Let m be the greatest integer such that $\phi_{k-j}(A_r) + \phi_{k-1}(A_s) = 1$ for all $j \in [\![m]\!]$. This implies that $\phi_{k-m-1}(A_r) = \phi_{k-m-1}(A_s) = 0$, since the definition of m gives that

$\phi_{k-m-1}(A_r)+\phi_{k-m-1}(A_s) \neq 1$ and the definition of k gives that $\phi_{k-m-1}(A_r)$ and $\phi_{k-m-1}(A_s)$ cannot be simultaneously 1.

$$\begin{aligned}
A_r + A_s &\subset \sum_{j=1}^{k-1} \big(\phi_j(A_r) + \phi_j(A_s)\big) V_j + V_{k-1} \\
&= \sum_{j=1}^{k-m-1} \big(\phi_j(A_r) + \phi_j(A_s)\big) V_j + \sum_{j=k-m}^{k-1} V_j + V_{k-1} \\
&\subset \sum_{j=1}^{k-m-1} \big(\phi_j(A_r) + \phi_j(A_s)\big) V_j + V_{k-m-1} \qquad \text{(by the lemma)} \\
&\subset \sum_{j=1}^{k-m-1} \phi_j(A_{r+s}) V_j + \sum_{j=k-m}^{\infty} \phi_j(A_{r+s}) V_j \\
&= A_{r+s}.
\end{aligned}$$

Our claim of subadditivity is shown.

We turn our attention to demonstrating properties of d:

(Translational invariance) Let $x, y, z \in X$. Then

$$\begin{aligned}
d(x+z, y+z) &= \pi\big((x+z) - (y+z)\big) \\
&= \pi(x-y)\big) \\
&= d(x, y)
\end{aligned}$$

It follows that any claim about $d(x, y)$ over X reduces to a claim about $d(\tilde{x}, 0)$ over X, and $d(\tilde{x}, 0)$ is just $\pi(\tilde{x})$.

(Positive definite) The range of π gives $d(x, y) = \pi(\tilde{x}) \geq 0$ for all $\tilde{x} = x - y \in X$. Since $0 \in A_r$ for all $r \in D$, $\pi(0) = \inf\{r \in D\} = 0$. For nonzero $x \in X$, there is some natural number k such that $4^k x \notin V_1$; since $x \in V_n$ implies $4x \in V_{n-1}$, which in turn implies that $4(4x) \in V_{n-2}$ and, inductively, $4^{n-1} \in V_1$, we have that $x \notin V_k$; it follows that $\pi(x) \geq 2^{-k} > 0$.

(Symmetric) Since each of the sets A_r is balanced, we have that $\pi^{-1}(r) = -\pi^{-1}(r)$, and thus $\pi(x) = \pi(-x)$ for all $x \in X$.

(Triangle inequality) Let $x, y, z \in X$. We would like to show that $d(x, z) \leq d(x, y) + d(y, z)$, which is equivalent to $\pi(x-z) \leq \pi(x-y) + \pi(y-z)$. Setting $a := x - y$ and $b := y - z$, and substituting, these are equivalent to

$$\pi(a+b) \leq \pi(a) + \pi(b).$$

If $\pi(a) + \pi(b) = 1$, there is nothing to show; assume $\pi(a) + \pi(b) = 1 - 2\epsilon$ for some $\epsilon > 0$. There exist $r, s \in D$ such that

$$r - \epsilon < \pi(a) < r \quad \text{and} \quad s - \epsilon < \pi(b) < s,$$

giving $r + s < 1$. We have that $a \in A_r$ and $b \in A_s$, and subadditivity in P yields $a + b \in A_r + A_s \subset A_{r+s}$, from which we ascertain that $\pi(a + b) \leq r + s$. Since the resulting inequalities

$$\pi(a + b) \leq r + s < \pi(a) + \pi(b) + 2\epsilon$$

hold true for any ϵ, we conclude that $\pi(a+b) \leq \pi(a)+\pi(b)$ and thus, d honors the triangle inequality.

Thus, d is a translation-invariant metric on X, and part (c) is proved. The open balls in τ_d centered at 0 are given by

$$B_\delta := \{x : \pi(x) < \delta\} = \bigcup_{\substack{r<\delta \\ r\in D}} A_r;$$

They are the union of sums of balanced the balanced sets V_n, and thus balanced, satisfying part (b) of the theorem. If X is locally convex, then the V_n's were chosen convex, and the open balls B_δ are convex, giving part (d).

Since $A_{2^{-n}} = V_n$ for all $n \in \mathbb{N}$, $B_\delta(0) \subset V_n$ for $\delta < 2^{-n}$, giving $\tau_d \subset \tau$. Given a $\delta > 0$, we may choose $n \in \mathbb{N}$ so that $\delta > 2^{-n}$, and the reverse inclusion yields $\tau = \tau_\delta$ by the compatible local bases and invariance, and the proof is complete. $\square$

3.0.4 Remark. It may be interesting to note that, while the elements of the local base $\{V_n\}$ were shown to be open in the topology induced by d, no V_n is a B_δ; consider, for example, V_1. The set $B_{1/2}(0) = \bigcup_{r<1/2} A_r$, so, dyadic $r < \frac{1}{2}$ has the property that $\phi_1(A_r) = 0$; it follows that

$$\begin{aligned}
\bigcup_{\substack{r<1/2 \\ r\in D}} A_r = \sum_{n=2}^{\infty} V_n & \\
&= V_2 + V_3 + \sum_{n=4}^{\infty} V_n \\
&\subset V_2 + V_3 + V_3 \\
&\subsetneq V_2 + V_2 + V_2 \\
&\subsetneq V_2 + V_2 + V_2 + V_2 \\
&\subset V_1
\end{aligned}$$

and $B_{1/2}(0)$ is a strict subset of V_1. For $\delta > \frac{1}{2}$, the density of D in $[0, 1]$ provides $r \in D$ such that $\frac{1}{2} < r < \delta$, and $V_1 \subsetneq B_\delta(0)$. A similar argument excludes V_n from $\{B_\delta(0)\}$ for all $n \in \mathbb{N}$.

References

[1] Munkres, James R., *Topology*, Prentice Hall, 2000