Functional Analysis, Math 7320
Lecture Notes from October 25, 2016

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1 Last time

Locally convex TVS as spaces whose topology is induced by a family of seminorms.

2 Warm-up

In a TVS, given $V \in \mathcal{U}$ there is $C \in \mathcal{U}$ s.t. $C \subset V$ and $C = \overline{C}$.

Proof. We know by continuity of addition at $(0,0)$ that, there is a $W \in \mathcal{U}$ with $W + W \subset V$. We also know that, $\overline{W} = \bigcap_{Y \in \mathcal{U}} (W+Y)$. So, by choosing $Y = W$, we have $\overline{W} \subset W+W \subset V$. □

3 Metrization

3.0.1 Question. When does the topology of a TVS come from a metric?

We have seen that in a TVS which comes from a metric, we have a countable local base, i.e. $(B_{1/n}(0))_{n\in\mathbb{N}}$. The existence of such a local base is also sufficient to obtain τ from a metric.

3.0.2 Theorem. *Let X be a TVS with a countable local base. Then, there is a metric d s.t.*

- ⓐ *Every open set in τ is the union of open balls w.r.t. d.*
- ⓑ *Each open $B_r(0)$, $r > 0$, is balanced,*
- ⓒ *d is translation invariant.*
- ⓓ *If X is locally convex, then d can be chosen so that all $B_r(0)$ are convex.*

Proof. Consider a countable local base B'. Let B be a countable local base s.t. $B = (V_n)_{n\in\mathbb{N}}$ with each V_n balanced and $V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subset V_n$ (using continuity of addition).
We then have for all $n, k \in \mathbb{N}$
$V_{n+1} + V_{n+2} + ... + V_{n+k} \subset V_n$ because $V_{n+k-1} + V_{n+k} \subset V_{n+k-1} + V_{n+k-1}$ and

$$\begin{aligned} V_{n+k-2} + V_{n+k-1} + V_{n+k} &\subset V_{n+k-2} + V_{n+k-1} + V_{n+k-1} \\ &\subset V_{n+k-2} + V_{n+k-2} + V_{n+k-2} \\ &\subset V_{n+k-3} \end{aligned}$$

We can iteratively remove tail terms and end up with

$$V_{n+1} + V_{n+2} + V_{n+2} + V_{n+2} \subset V_{n+1} + V_{n+1} \subset V_n$$

Consider the dyadic rationals $D = \{\sum_{j=1}^{n} \frac{c_j}{2^j},\ c_j \in \{0,1\}, n \in \mathbb{N}\}$ and let

$\phi : D \cup [1, \infty] \to P(X)$

$$\phi(r) = \begin{cases} X & r \geq 1 \\ c_1(r)V_1 + c_2(r)V_2 + ... + c_n(r)V_n, & r \in D \end{cases}$$

From the construction of $(V_n)_{n\in\mathbb{N}}$,

$$\phi(\sum_{j=n_1}^{n_2} \frac{c_j}{2^j}) = \sum_{j=n_1}^{n_2} c_j V_j \subset V_{n_1-1}.$$

Further, let $f : X \to \mathbb{R}$

$$f(x) = \inf\{r : x \in \phi(r)\}$$

and let $d(x, y) = f(y - x)$.

We first consider properties of ϕ. For $r, s \in D$, we claim that $\phi(r) + \phi(s) \subset \phi(r + s)$.

(i) If $r + s \geq 1$ then the RHS gives $\phi(r + s) = X$, and there is nothing to show.

(ii) Next, let $r + s \in D$.

(Case I) $c_n(r) + c_n(s) = c_n(r + s)$ for all n. Then

$$\begin{aligned} \phi(r + s) &= \sum_{j=1}^{N} c_j(r + s)V_j \\ &= \sum_{j=1}^{N} c_j(r)V_j + \sum_{j=1}^{N} c_j(s)V_j \\ &= \phi(r) + \phi(s) \end{aligned}$$

(Case II) There is an $n \in \mathbb{N}$ s.t. $c_n(r) + c_n(s) \neq c_n(r + s)$. Let N be the smallest index for which this occurs, then $c_N(r) = c_N(s) = 0$, and $c_N(r + s) = 1$. Consequently,

$$\begin{aligned} \phi(r) = & c_1(r)V_1 + c_2(r)V_2 + ... + c_{N-1}(r)V_{N-1} + 0.V_N \\ & + c_{N+1}(r)V_{N+1} + c_{N+2}(r)V_{N+2} + c_{N+3}(r)V_{N+3} + ... \\ & \subset c_1(r)V_1 + ... + c_{N-1}(r)V_{N-1} + V_{N+1} + V_{N+1} \end{aligned}$$

and similarly
$\phi(s) \subset c_1(s)V_1 + ... + c_{N-1}(s)V_{N-1} + V_{N+1} + V_{N+1}$ Hence,

$$\begin{aligned}\phi(r) + \phi(s) &\subset c_1(r+s)V_1 + ... + c_{N-1}(r+s)V_{N-1} + V_{N+1} + V_{N+1} + V_{N+1} + V_{N+1} \\ &\subset c_1(r+s)V_1 + ... + c_{N-1}(r+s)V_{N-1} + \underbrace{c_N(r+s)}_{1}V_N \\ &\subset \phi(r+s)\end{aligned}$$

Next, we observe that for $r \in D \cup [1, \infty)$, $0 \in \phi(r)$, because $\phi(r)$ contains at least one neighborhood of 0. Moreover, $\{\phi(r) : r \in D\}$ is totally ordered by set inclusion, because if $r < t$, then $\phi(r) \subset \phi(r) + \phi(t-r) \subset \phi(t)$. From the definition of f, this implies for all $x, y \in X$, $f(x+y) \leq f(x) + f(y)$, as we see below:

From the range of f being $[0,1]$, assume $RHS \lneq 1$:

Fix $\epsilon > 0$, then there are $r, s \in D$ with $f(x) < r, f(y) < s$ and $r+s < f(x)+f(y)+\epsilon$. Ordering implies $x \in \phi(r), y \in \phi(s)$ and therefore $(x+y) \in \phi(r) + \phi(s) \subset \phi(r+s)$. Therefore $f(x+y) \leq r+s < f(x)+f(y)+\epsilon$ This is true for any $\epsilon > 0$, so $f(x+y) \leq f(x) + f(y)$.

Next, by balancedness of each V_j, $f(x) = f(-x)$, $f(0) = 0$, and $f(x) > 0$ for $x \neq 0$.

This results from each $\phi(r)$ being balanced, and $f(0) = 0$ because $0 \in \phi(r)$ for each $r \in D$, and if $x \neq 0$, then there is V_N with $x \notin V_N$ (since every TVS is a Hausdorff space, so we can separate x from 0 by two disjoint open neighborhoods), since by construction $V_k \supset V_N$ for all $k \leq N$ therefore, there is s s.t. $x \notin \phi(r)$ for all $r < s$, and so by definition $f(x) \gneq 0$.

We conclude that $d(x,y) = f(x-y)$ defines a (translationally) invariant metric. To see that d is compatible with τ, consider

$$\begin{aligned}B_\delta(0) &= \{x : f(x) < \delta\} \\ &= \text{(by total ordering)} \\ &= \bigcup_{\substack{r<\delta \\ r\in D}} \phi(r)\end{aligned}$$

Thus, $B_{\frac{1}{2^n}}(0)$ is a local base.

If each V_n is convex, so is each $\phi(r)$ and hence $B_{\frac{1}{2^n}}(0)$

$\square$

3.0.3 Theorem. *Suppose that (X, d_1) and (Y, d_2) are metric spaces, and (X, d_1) is complete. If E is closed in X, $f : E \to Y$ is continuous, and*

$$d_2(f(x'), f(x'')) \geq d_1(x', x'') \tag{*}$$

for all $x', x'' \in E$, then $f(E)$ is closed.

Proof. Pick $y \in \overline{f(E)}$. There exist points $x_n \in E$ so that $y = \lim f(x_n)$. Thus $\{f(x_n)\}$ is Cauchy in Y. Our hypothesis $(*)$ implies therefore that $\{x_n\}$ is Cauchy in X. Being a closed subset of a complete metric space, E is complete; Hence there exists $x = \lim x_n$ in E. Since f is continuous, $f(x) = \lim f(x_n) = y$.

Thus $y \in f(E)$ □

3.0.4 Theorem. (a) *If d is a translation-invariant metric on a v.s. X, then*

$$d(nx, 0) \leq nd(x, 0)$$

(b) *If $\{x_n\}$ is a sequence in a metrizable t.v.s. X and if $x_n \to 0$ as $n \to \infty$, then there are positive scalars γ_n s.t. $\gamma_n \to \infty$ and $\gamma_n x_n \to 0$.*

Proof. Statement (a) follows from triangle inequality plus translation invariance of the metric

$$\begin{aligned} d(0, nx) &\leq d(0, x) + d(x, 2x) + d(2x, 3x) + ... + d((n-1)x, nx) \\ &\leq \sum_{k=1}^{n} d(kx, (k-1)x) \\ &= \underbrace{d(x, 0) + d(x, 0) + d(x, 0) + ... + d(x, 0)}_{n\_\text{times}} \\ &= nd(x, 0) \end{aligned}$$

To prove (b), let d be a metric as in (a), compatible with the topology of X. Since $d(x_n, 0) \to 0$, there is an increasing sequence of positive integers n_k such that $d(x_n, 0) < \frac{1}{k^2}$ if $n \geq n_k$. Put $\begin{cases} \gamma_n = 1 & \text{if } n < n_1, \\ \gamma_n = k & \text{if } n_k \leq n < n_{k+1}, \end{cases}$ for such n we have

$d(\gamma_n x_n, 0) = d(kx_n, 0) \leq kd(x_n, 0) < \frac{1}{k}$. Hence $\gamma_n x_n \to 0$ as $n \to \infty$. □