Functional Analysis, Math 7320
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3 Completeness

What is the Cauchy property in a topological vector space? Can we define "completeness" in this setting? Although we do not necessarily have a metric space structure, we can still define these concepts in a way that makes sense. Note that even though we *can* define a notion of completeness, we do not explicitly use this definition in the following discussion.

3.0.1 Definition. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is *Cauchy* if for each $U \in \mathcal{U}$ there exists an $N \in \mathbb{N}$ such that for all $n, m \geq \mathbb{N}$, $x_n - x_m \in U$. A TVS X is called *complete* if every Cauchy sequence converges to some $x \in X$.

Similar to how null sets are considered "small" from the perspective of measure theory, we introduce the Baire categories as a way to talk about topologically small/large sets in a general topological space. We will then examine this concept in the TVS setting by determining when a TVS is a Baire space, and formulate some powerful results using the language of category.

3.0.2 Definition. Let X be a topological space.

(a) A subset $E \subset X$ is called *nowhere dense* if $(\overline{E})^\circ = \emptyset$, or equivalently if $X \setminus \overline{E}$ is dense in X.

(b) A subset $E \subset X$ is said to be of *first category* in X if $E = \bigcup_{n=1}^\infty E_n$ with each E_n nowhere dense. Otherwise, E is said to be of *second category*.

(c) X is called a *Baire space* if for each sequence $(U_n)_{n\in\mathbb{N}}$ of open dense sets in X, $\bigcap_{n=1}^\infty U_n$ is dense in X.

3.0.3 Remark. Let X be a Baire space. For any countable collection $(E_n)_{n\in\mathbb{N}}$ of nowhere dense subsets of X, let $V_n = X \setminus \overline{E_n}$ (note each V_n is dense in X). Then by definition of a Baire space $\bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$, hence $\bigcup_{n\in\mathbb{N}} E_n \subset \bigcup_{n\in\mathbb{N}} \overline{E_n} = X \setminus \bigcap_{n\in\mathbb{N}} V_n \subsetneq X$. In other words, Baire spaces are of second category in themselves.

We next note some simple properties of category.

3.0.4 Lemma. *Let X be a topological space.*

1. *If $A \subset B \subset X$, and B is of first category, so is A.*

2. *If each E_n is of first category, so is $\bigcup_{n=1}^{\infty} E_n$.*

3. *If $E \subset X$ is closed and $E^\circ = \emptyset$, then E is of first category.*

Proof. For (1), note that if B is of first category then $B = \bigcup_{n\in\mathbb{N}} E_n$ where each E_n is nowhere dense. Since $A \subset B$, we have $A = \bigcup_{n\in\mathbb{N}}(A \cap E_n)$, and clearly each $A \cap E_n$ is nowhere dense since it is a subset of E_n. Thus A is of first category.

For (2), suppose E_n is of first category for all $n \in \mathbb{N}$, so $E_n = \bigcup_{j\in\mathbb{N}} E_{n,j}$ with each $E_{n,j}$ nowhere dense. Then $E = \bigcup_{n\in\mathbb{N}} E_n$ is a countable union of nowhere dense sets, hence E is of first category.

Lastly, for (3) note that if E is closed, then $E = \overline{E}$. So $\emptyset = E^\circ = (\overline{E})^\circ$, hence E is nowhere dense. □

Our next lemma highlights one of the important characterizations of a Baire space, namely that the countable union of closed sets with empty interior must itself have empty interior.

3.0.5 Lemma. *Let X be a topological space. Then X is a Baire space if and only if for each $(F_n)_{n\in\mathbb{N}}$ with each F_n closed and $(\bigcup_{n\in\mathbb{N}} F_n)^\circ \neq \emptyset$, at least one F_n has $F_n^\circ \neq \emptyset$.*

Proof. Let X be a Baire space and $(F_n)_{n\in\mathbb{N}}$ a sequence of closed sets with and suppose that for each $n \in \mathbb{N}$, $F_n^\circ = \emptyset$. Then for each $n \in \mathbb{N}$ we have $X \setminus F_n^\circ = X$, so $\overline{X \setminus F_n} = X$. So each $X \setminus F_n$ is open and dense in X. Since X is a Baire space, we know:

$$\begin{aligned}
\overline{\bigcap_{n\in\mathbb{N}}(X \setminus F_n)} = X \implies& \overline{X \setminus \bigcup_{n\in\mathbb{N}} F_n} = X \\
\implies& X \setminus \left(\bigcup_{n\in\mathbb{N}} F_n\right)^\circ = X \\
\implies& \left(\bigcup_{n\in\mathbb{N}} F_n\right)^\circ = \emptyset.
\end{aligned}$$

Conversely, assume that for each sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets with $(\bigcup_{n\in\mathbb{N}} F_n)^\circ \neq \emptyset$, then at least one F_n satisfies $F_n^\circ \neq \emptyset$. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open dense sets in X. Then $F_n = X \setminus U_n$ is closed and $F_n^\circ = \emptyset$, so $(\bigcup_{n\in\mathbb{N}} F_n)^\circ = \emptyset$, hence $\overline{(\bigcap_{n\in\mathbb{N}} U_n)} = X$. So X is a Baire space. □

We now give two sufficient conditions for a topological vector space to be a Baire space.

3.0.6 Theorem. *A topological vector space X is a Baire space if:*

1. *X is locally compact.*

2. *The topology on X is induced by a complete metric.*

Proof. Assume $(V_n)_{n\in\mathbb{N}}$ are open dense subsets of X. Let $B_0 \neq \emptyset$ be open, and for each $n \in \mathbb{N}$ choose $B_{n+1} \neq \emptyset$ such that $\overline{B_{n+1}} \subset V_{n+1} \cap B_n$.

Suppose now that X is locally compact. Then we can choose each B_n such that $\overline{B_n}$ is compact, so $K = \bigcap_{n\in\mathbb{N}} \overline{B_n} \neq \emptyset$ by compactness (and the face that the B_n's are nested). So $\emptyset \neq K \subset \bigcap_{n\in\mathbb{N}} V_n$, thus X is a Baire space.

Next, suppose (2) is true. Then we can choose each B_n to be a ball of radius $1/n$. Let $x_n \in B_n$ for all $n \in \mathbb{N}$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence due to the fact the B_n's are nested with radii approaching zero. Thus (x_n) converges to some $x \in X$ since the topology of X is induced by a complete metric. So x is a limit point of $\bigcap_{n\in\mathbb{N}} \overline{B_n}$, hence $x \in \bigcap_{n\in\mathbb{N}} \overline{B_n} \subset \bigcap_{n\in\mathbb{N}} V_n$. Thus X is a Baire space. □

Our next major use of the Baire categories is in formulating and proving a general version of the Banach-Steinhaus theorem. To do so, we first define the notions of equicontinuity and uniform boundedness for a family of maps, and show how they are related.

3.0.7 Definition. Let X and Y be topological vector spaces, and Γ a collection of linear maps from X to Y.

(a) We say that Γ is *equicontinuous* if for each $V \in \mathcal{U}^Y$, there exists a $W \in \mathcal{U}^X$ such that for all $A \in \Gamma$, $A(W) \subset V$.

(b) We say that Γ is *uniformly bounded* if for each bounded set $E \subset X$ there is a bounded set $F \subset Y$ such that for all $A \in \Gamma$, $A(E) \subset F$.

3.0.8 Proposition. *Let X and Y be topological vector spaces, and Γ an equicontinuous collection of linear maps from X to Y. Then Γ is uniformly bounded.*

Proof. Let $E \subset X$ be bounded, and set $F = \bigcup_{A\in\Gamma} A(E)$. We want to show that F is bounded. Consider any $V \in \mathcal{U}^Y$. By equicontinuity, there exists a $W \in \mathcal{U}^X$ such that for each $A \in \Gamma$, $A(W) \subset V$. By boundedness of W there exists some $t_0 > 0$ such that $E \subset tW$ for all $t > t_0$. Thus for any $A \in \Gamma$, $A(E) \subset A(tW) = tA(W) \subset tV$ for all $t > t_0$. Thus $A(E)$ is bounded, and moreover $F \subset tV$ so F is bounded. □

Finally, we can show the Banach-Steinhaus theorem. This theorem says that if a large enough (in terms of category) subset of a TVS X has bounded orbit under a family of continuous linear maps, then the family is actually equicontinuous.

3.0.9 Theorem. *(Banach-Steinhaus) Let X and Y be topological vector spaces, and Γ a collection of continuous linear maps from X to Y. If*

$$B := \big\{x \in X : \{Ax : A \in \Gamma\} \text{ is bounded}\big\}$$

is of second category, then $B = X$ and Γ is equicontinuous (and hence also uniformly bounded).

Proof. Let U, W be balanced neighborhoods of $0 \in Y$ such that $\overline{U} + \overline{U} \subset W$, and let $E = \bigcap_{A\in\Gamma} A^{-1}(\overline{U})$. Then E is closed by continuity of each $A \in \Gamma$, and in particular we have $E = \{x \in X : A(x) \in \overline{U} \text{ for all } A \in \Gamma\}$.

For a given $x \in B$, the set $\{Ax : A \in \Gamma\}$ is bounded, so there exists some $n \in \mathbb{N}$ with $\{Ax : A \in \Gamma\} \subset nU \subset n\overline{U}$. Hence $x \in nE$. So $B \subset \bigcup_{n=1}^{\infty} nE$, and since B is of second category it follows that nE is of second category for at least one $n \in \mathbb{N}$. Since scalar multiplication is a homeomorphism, this means E itself is of second category.

By E being closed and of second category, there exists an interior point $x \in E^\circ$, and so $E - x$ contains a neighborhood V of 0. Note that $V \subset E - x$ satisfies:

$$A(V) \subset A(E) - Ax \subset \overline{U} - \overline{U} = \overline{U} + \overline{U} \subset W$$

for all $A \in \Gamma$. Thus Γ is equicontinuous, so Γ is uniformly bounded which means $B = X$. □