Functional Analysis, Math 7320
Lecture Notes from October 27, 2016

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Warm up: Could we define completeness for topological vector spaces? What is the Cauchy property in a topological vector space?

2.9.9 Definition. A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological vector space X is called Cauchy if for each $U \in \mathcal{U}$, there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $x_n - x_m \in U$.

So, we say a topological vector space X is **complete** if every Cauchy sequence converges.

3 Completeness

3.1 Baire categories

3.1.1 Definition. (a) Let X be a topological vector space. A subset $E \subset X$ is called nowhere dense if $\overline{E}^\circ = \emptyset$, or equivalently, if $X \setminus \overline{E}$ is dense in X.

(b) A subset $E \subset X$ is of first category in X if $E = \cup_{n=1}^\infty E_n$ with each E_n nowhere dense. Otherwise, E is of the second category.

(c) X is called a Baire space if for each sequence $(U_n)_{n\in\mathbb{N}}$ of open, dense sets in X, then $\cap_{n=1}^\infty U_n$ is dense in X.

3.1.2 Remark. It is worth noticing that if E can be written as a countable union as above, then it is of second category if at least one of the component sets is **not** nowhere dense.

3.1.3 Examples. (a) The set $\{1/n : n \in \mathbb{N}\}$ is nowhere dense in $\mathbb{R}$, since although its points get arbitrarily close to 0, the closure $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ has empty interior.

(b) The rational numbers are of first category as a subset of the reals but also as a space, which means they do not form a Baire space.

(c) The Cantor set is of first category as a subset of the reals, but as a space, it is a complete metric space and is thus a Baire space, as we will see from the Baire category theorem in what follows.

Next we formulate elementary properties.

3.1.4 Lemma. *Let X be a topological vector space.*

(a) *If each $E_n \subset X$, $n \in \mathbb{N}$ is of first category, then so is $\cup_{n=1}^{\infty} E_n$.*

(b) *If $E \subset X$ is closed and $E^\circ = \emptyset$, then E is of first category.*

3.1.5 Lemma. *Let X be a topological vector space. Then X is a Baire space if and only if for each sequence $(F_n)_{n\in\mathbb{N}}$ with F_n closed and $(\cup_{n=1}^{\infty} F_n)^\circ \neq \emptyset$, then $F_n^\circ \neq \emptyset$ for at least one F_n.*

Proof. Let X be a Baire space and let $(F_n)_{n\in\mathbb{N}}$ be a sequence of closed sets in X. Moreover, assume $F_n^\circ = \emptyset$ for all $n \in \mathbb{N}$, or equivalently $X \setminus (F_n)^\circ = X$. This implies

$$\overline{(X \setminus F_n)} = X,$$

and since X is a Baire space, we have

$$X = \overline{\left(\bigcap_{n\in\mathbb{N}} (X \setminus F_n)\right)} = \overline{\left(X \setminus \bigcup_{n\in\mathbb{N}} F_n\right)},$$

which means that $(\cup_{n\in\mathbb{N}} F_n)^\circ = \emptyset$. Thus, if $(\cup_{n\in\mathbb{N}} F_n)^\circ \neq \emptyset$, then it must be that at least one F_n has nonempty interior.

Conversely, given $(U_n)_{n\in\mathbb{N}}$ of open dense sets, then $F_n = X \setminus U_n$ defines a sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets with $F_n^\circ = \emptyset$ and so $(\cup_{n\in\mathbb{N}} F_n)^\circ = \emptyset$, which in turn gives $\overline{(\cap_{n\in\mathbb{N}} U_n)} = X$, by De Morgan and hence X is by definition a Baire space. $\square$

At this stage, and before stating the Baire category theorem, we state two variants of Cantor's intersection theorems.

3.1.6 Theorem. *(Cantor's intersection theorem) A decreasing nested sequence of non-empty compact subsets of a compact topological space X has non-empty intersection. In other words, if $(B_n)_{n\in\mathbb{N}}$ is a sequence of non-empty compact subsets of X satisfying*

$$B_1 \supset B_2 \supset \ldots \supset B_k \supset B_{k+1} \supset \ldots,$$

then

$$\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset.$$

Proof. Suppose that $\cap_{n\in\mathbb{N}} B_n = \emptyset$ and let $V_n = X \setminus B_n$ for all $n \in \mathbb{N}$. Note that $(V_n)_{n\in\mathbb{N}}$ is an open cover of X, as

$$X = X \setminus \bigcap_{n\in\mathbb{N}} V_n = \bigcup_{n\in\mathbb{N}} (X \setminus V_n) = \bigcup_{n\in\mathbb{N}} V_n.$$

We extract a finite subcover of X and since $V_1 \subset V_2 \subset \ldots$, there must exist a $k \in \mathbb{N}$ such that $V_k = X$. Then $B_k = X \setminus V_k = \emptyset$, contradicting the non-emptiness of all the B_n's. Thus the intersection is non-empty. $\square$

3.1.7 Theorem. *(Cantor's intersection theorem for complete metric spaces) Let X be a complete metric space, and let $(B_n)_{n\in\mathbb{N}}$ be a decreasing nested sequence of non-empty closed subsets of X, with diam $B_n \to 0$. Then $\cap_{n\in\mathbb{N}} B_n \neq \emptyset$.*

Proof. In each set B_n, we choose a point x_n. Then the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy since if $m, n \geq N$, we have $d(x_m, x_n) \leq$ diam B_n, which tends to zero as $N \to \infty$. Since X is complete, the sequence $(x_n)_{n\in\mathbb{N}}$ has a limit x. But since $x_n \in B_n$ for all $n \geq N$, and B_n is closed, we have $x \in B_n$. This holds for all n and so we have $x \in \cap_{n\in\mathbb{N}} B_n$. □

We are now ready to state and prove the Baire category theorem.

3.1.8 Theorem. *(Baire category theorem) A topological vector space X is a Baire space if*

1. *X is locally compact.*

2. *The topology on X is induced by a complete metric.*

Proof.

1. Let X be a locally compact topological vector space and let $V_1, V_2, \ldots$ be dense, open subsets of X. Also, let $B_0 \neq \emptyset$ be an open subset of X. We want to show that $\cap_{n\in\mathbb{N}} V_n$ is dense in X, or equivalently, that it intersects B_0.

   To this end, we notice that $B_0 \cap V_1$ is open in X, and non-empty, as B_0 is open and V_1 is dense. So we can find an open set B_1 with compact closure such that $\overline{B_1} \subset B_0 \cap V_1$, by local compactness and Hausdorff. Now $B_1 \cap V_2$ is open and non-empty as above, and so again, we can find an open and non-empty B_2 with compact closure such that $\overline{B_2} \subset B_1 \cap V_2$ (we notice that $\overline{B_2} \subset B_0 \cap V_1 \cap V_2$ as well).

   In this recursive manner, we choose a sequence $(B_n)_{n\in\mathbb{N}}$ of non-empty open subsets of X with $\overline{B_n}$ compact for all n, such that for each $n \in \mathbb{N}$, $\overline{B_{n+1}} \subset V_{n+1} \cap B_n$. We then notice that $(\overline{B_n})_{n\in\mathbb{N}}$ satisfies the assumptions of Cantor's intersection theorem and so $K = \cap_{n\in\mathbb{N}} \overline{B_n} \neq \emptyset$. Since $K \subset \cap_{n\in\mathbb{N}} V_n$, we have $B_0 \cap (\cap_{n\in\mathbb{N}} V_n) \neq \emptyset$, which is precisely what we aimed for.

2. Assuming the topology on X is induced by a complete metric and in the light of the proof in part (1), we now choose B_n, $n \in \mathbb{N}$, to be an open ball of radius $1/n$ and obtain $\cap_{n\in\mathbb{N}} \overline{B_n} \neq \emptyset$, this time using Cantor's intersection theorem for complete spaces.

□

3.2 Uniform boundedness

We first show that uniform boundedness is a consequence of equicontinuity.

3.2.9 Definition. Let X, Y be topological vector spaces and let Γ be a collection of linear maps from X to Y. We say that Γ is equicontinuous if for each $V \in \mathcal{U}^Y$ (neighborhood of zero in Y), there is $W \in \mathcal{U}^X$ such that for all $A \in \Gamma$, $A(W) \subset V$.

Next we see how equicontinuity implies uniform boundedness.

3.2.10 Proposition. *Let X, Y be topological vector spaces and let Γ be an equicontinuous collection of linear maps from X to Y. Then, for each bounded set $E \subset X$, there is a bounded set $F \subset Y$ such that for all $A \in \Gamma$, $A(E) \subset F$.*

Proof. Let $E \subset X$ be bounded and set $F = \cup_{A\in\Gamma} A(E)$. It suffices to show F is bounded. To this end, we consider any $V \in \mathcal{U}^Y$ and notice that by equicontinuity, there is $W \in \mathcal{U}^X$ such that for each $A \in \Gamma$, $A(W) \subset V$. By the boundedness of E, there is $s > 0$ with $E \subset tW$, for all $t > s$. Consequently, for all $A \in \Gamma$ and for all $t > s$,

$$A(E) \subset A(tW) = tA(W) \subset tV,$$

and thus, by taking unions, $F \subset tV$, which means F is bounded. □

Next, we state the Banach-Steinhaus theorem.

3.2.11 Theorem. *(Banach-Steinhaus) Let X, Y be topological vector spaces and let Γ be a collection of continuous linear maps from X to Y. If*

$$B = \{x \in X : \{Ax : x \in \Gamma\} \text{ is bounded}\}$$

is of second category, then $B = X$ and Γ is equicontinuous.

Proof. By our assumption, B is not the countable union of nowhere dense sets. Let $\mathcal{U}, \mathcal{W}$ be balanced neighborhoods of zero in Y such that $\overline{\mathcal{U}} + \overline{\mathcal{U}} \subset \mathcal{W}$, and let

$$\begin{aligned} E &= \bigcap_{A\in\Gamma} A^{-1}(\overline{\mathcal{U}}) \\ &= \{x \in X : \text{ for all } A \in \Gamma, Ax \in \overline{\mathcal{U}}\}. \end{aligned}$$

We notice that by continuity of each A, E is closed. Now by the boundedness assumption, for a given $x \in B$, there is $n \in \mathbb{N}$ with

$$\{Ax : A \in \Gamma\} \subset n\mathcal{U} \subset n\overline{\mathcal{U}},$$

and by comparison with the definition of E and scaling, $x \in nE$. Thus, $B \subset \cup_{n\in\mathbb{N}} nE$. From B being of second category, at least one of the nE's is of second category and so E is of second category (since scaling by n is a homeomorphism). Hence, since $E = \overline{E}$, there is an interior point, say $x \in X$, and $E - x$ contains a neighborhood $V \in \mathcal{U}$. Consequently, for each $A \in \Gamma$, $V \subset E - x$ satisfies

$$A(V) \subset A(E) - Ax \subset \overline{\mathcal{U}} - \overline{\mathcal{U}} = \overline{\mathcal{U}} + \overline{\mathcal{U}} \subset \mathcal{W},$$

since $\mathcal{U}$ is balanced. Thus, Γ is equicontinuous. Finally, from the preceding proposition, we obtain that Γ is uniformly bounded and hence $B = X$. □

The following version of the Banach-Steinhaus theorem is called uniform boundedness principle.

3.2.12 Theorem. *(Uniform boundedness principle) Let Γ be a family of continuous linear mappings from a Banach space X into a normed space Y. If for every $x \in X$,*

$$\sup_{A\in\Gamma} \|Ax\| < \infty,$$

then $\sup_{A\in\Gamma} \|\Gamma\| < \infty$.

Proof. Let $E = \{x \in X : \sup_{A\in\Gamma} \|Ax\| \leq 1\}$. Since

$$X = \bigcup_{n\in\mathbb{N}} nE$$

and X is complete, Baire's theorem implies that E has an interior point x_0. Then $E - x_0$ is a neighborhood of 0 and so there exists $\epsilon > 0$ so that $B_\epsilon(0) \subset E - x_0$. Hence, for any $0 \neq x \in B_\epsilon(0)$ and $A \in \Gamma$, we have

$$\|Ax\| \leq \|A(x + x_0)\| + \|Ax_0\| \leq 1 + \sup_{A\in\Gamma} \|Ax_0\|.$$

Thus, we have

$$\sup_{A\in\Gamma} \|A\| \leq \frac{1 + \sup_{A\in\Gamma} \|Ax_0\|}{\epsilon}.$$

□

3.2.13 Example. An important implication of the Banach-Steinhaus theorem is the divergence of Fourier series of C^0 functions. The general discussion of L^2 functions shows that a Cauchy sequence of L^2 functions has a subsequence converging pointwise. Indeed, this proves the existence of the limit when we prove completeness of L^2. Moreover, this applies to Fourier series, but does not say anything about the pointwise convergence of the whole sequence of partial sums, and does not address uniformity of the pointwise convergence. Specifically, there is $f \in C^0(S^1)$ whose Fourier series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}, \qquad \text{with } \hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)dx,$$

diverges at zero.

Proof. To set our way towards satisfying the Banach-Steinhaus premises, consider the functionals given by the partial sums of the Fourier series of f, evaluated at 0, i.e.,

$$T_N(f) = \sum_{|n|\leq N} \hat{f}(n).$$

Then, we can get an upper bound by

$$\begin{aligned} |T_N(f)| &\leq \frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{|n|\leq N} e^{-inx} \right| |f(x)|dx \\ &\leq \|f\|_{C^0} \frac{1}{2\pi}\int_0^{2\pi} |D_N(x)|dx \\ &= \|f\|_{C^0} \|D_N\|_{L^1(S^1)}, \end{aligned}$$

where D_N denotes the Dirichlet kernel $\sum_{|n|\leq N} e^{-inx}$. Next, we aim to show that in the above inequality what actually holds is equality, or equivalently, that $|T_N| = \|D_N\|_{L^1(S^1)}$, but also that

$\|D_N\|_{L^1(S^1)} \to \infty$, as $N \to \infty$. In doing so, we notice that by summing the finite geometric series we obtain

$$D_N(x) = \frac{\sin\left(N + \frac{1}{2}\right) x}{\sin\left(\frac{x}{2}\right)},$$

while since $|\sin(t)| \leq |t|$, we can get a lower bound by

$$\begin{aligned}
\int_0^{2\pi} |D_N(x)| dx &\geq \int_0^{2\pi} \left|\sin\left(N + \frac{1}{2}\right) x\right| \frac{2}{x} dx \\
&= \int_0^{2\pi\left(N+\frac{1}{2}\right)} |\sin(x)| \frac{2}{x} dx \\
&\geq \sum_{k=1}^{N} \frac{1}{k} \int_{2\pi(k-1)}^{2\pi k} |\sin(x)| dx \\
&\geq \sum_{k=1}^{N} \frac{1}{k} \to \infty,
\end{aligned}$$

as $N \to \infty$. Thus, the L^1 norms diverge. Now let $g(x) = \text{sign } D_N(x)$ and let g_j be a sequence of periodic functions with $|g_j| \leq 1$ and going to g pointwise. Then, by dominated convergence,

$$\begin{aligned}
\lim_{j\to\infty} T_N(g_j) &= \lim_{j\to\infty} \int_0^{2\pi} g_j(x) D_N(x) dx \\
&= \int_0^{2\pi} g(x) D_N(x) dx \\
&= \int_0^{2\pi} |D_N(x)| dx.
\end{aligned}$$

Lastly, by Banach-Steinhaus for the Banach space $C^0(S^1)$, since there is no uniform bound $M > 0$ such that $|T_N| \leq M$ for all N, there exists f in the unit ball of $C^0(S^1)$ such that

$$\sup_N |T_N f| = \infty.$$

In fact, the collection of such functions f is dense in the unit ball, and is a G_δ set. That is, the Fourier series of f does not converge at 0. $\square$