Functional Analysis, Math 7320
Lecture Notes from November 1st, 2016

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3.2.14 Corollary. *Let X be F-spaces, Y be topological vector spaces and let Γ be a collection of continuous linear maps from X to Y. If*

$$B_x = \{Ax : A \in \Gamma\}$$

is bounded in Y, for every $x \in X$. Then Γ is equicontinuous and hence uniformly bounded.

Proof. here we have

$$B = \{x \in X : B_x \text{ is bounded}\} = X$$

is of second category, so the preceding theorem applies. □

we specialize further to norm space X, Y.

3.2.15 Theorem. *(Banach-Steinhaus) Let Γ be a family of continuous linear mappings from a Banach space X into a normed space Y. If for every $x \in X$,*

$$\sup_{A \in \Gamma} \|Ax\| < \infty,$$

then there exists $M > 0$ s.t for all $A \in \Gamma, x \in X, \|x\| \le 1$, we have

$$|Ax| \le M.$$

Hence

$$\sup_{A \in \Gamma} \|A\| \le M.$$

we investigate consequence of sequence of operators:

Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of continuous map from F-space to Topological vector space Y, and for each $x \in X$,

$$\lim_{n \to \infty} A_n(x) = A(x) \text{ exists.}$$

Question: is A bounded?

3.2.16 Corollary. *Let X be F-spaces, Y be topological vector spaces and let $(A_n)_{n \in \mathbb{N}}$ be a collection of continuous linear maps from X to Y. And suppose for each $x \in X$,*

$$\lim_{n \to \infty} A_n(x) = A(x) \text{ exists.}$$

Then A is linear and bounded.

Proof. let $x \in X$, by the condition of $(A_n x)_{n\in\mathbb{N}}$ in Y, $(A_n x)_{n\in\mathbb{N}}$ is bounded in Y, so $(A_n)_{n\in\mathbb{N}}$ is equicontinuous by Banach-Steinhaus for F-space.

Thus if $W \in \mathcal{U}^Y$, then there is $V \in \mathcal{U}^X$, s.t

$$A_n(V) \subset W \text{ for each } n \in \mathbb{N}.$$

then

$$\left\{y \in Y : y = \lim_{n\to\infty} A_n(x) \text{ for } x \in V\right\} \subset \overline{W}.$$

so

$$A(V) \subset \overline{W}.$$

Hence A is bounded. □

3.2.17 Example. Consider c_{00}, and let $A_n : c_{00} \to c_{00}$ be given by:

$$(A_n x)_m = \begin{cases} m x_m & \text{if } m \leq n \\ 0 & \text{if } m > n \end{cases} \tag{1}$$

then

$$\|A_n\| = n < \infty.$$

So each A_n is bounded and for each $x \in c_{00}$,

$$\lim_{n\to\infty} A_n(x) = A(x) \text{ exists.}$$

But, choosing $e_m = (0, 0 \ldots, 0, 1, 0, \ldots)$ where 1 is in the mth position, shows:

$$\sup_{\|x\|_\infty \leq 1, x\in c_{00}} \|Ax\| > \sup_{m\in\mathbb{N}} |Ae_m| = \sup_{m\in\infty} m = \infty.$$

This is not a contradiction to uniform boundedness, because c_{00} is not complete. If we replace c_{00} with c_0, then however with

$$x = (n^{-1/2})_{n\in\mathbb{N}} \in c_0,$$

we would have

$$(Ax)_n = n^{1/2} \to \infty.$$

so

$$Ax \notin c_0,$$

that means we lose pointwise convention.

3.3 Open mapping theorem

we recall that if X, Y are Hausdorff, X is compact and $f : X \to Y$ onto continuous, then f is open map. Now we define an analogous result for maps between F-space.

3.3.18 Theorem. *Let A be a continuous map from F-space X to a TVS space Y which is continuous, linear and $A(X)$ is of second category in Y, then $A(X) = Y$, A is open and Y is an F-space.*

To prove that A is open, we only need to show if it is open at 0, i.e an open neighborhood of $0 \in X$ is mapped to a open neighborhood of $0 \in Y$. After proving this, we have for any balanced $V \in \mathcal{U}^X$, there is a balanced $W \in \mathcal{U}^Y$ such that

$$W \subset A(V) \subset A(X).$$

but by linearity, for any $n \in \mathbb{N}$,

$$nW \subset nA(V) \subset A(X).$$

Next, we want to show $\overline{A(V_2)}$ has non-empty interior to get $W \in \mathcal{U}^Y$ with $W \subset \overline{A(V_1)}$.

Next,

$$A(X) = A(\bigcup_{k=1}^{\infty} kV_2) = \bigcup_{k=1}^{\infty}(kA(V_2)).$$

So one of $kA(V_2)$ is of second category, but scaling with M_k is a homeomorphism, so $A(V_2)$ is of second category, so $\overline{A(V_2)}$.

We still need to show $\overline{A(V_1)} \subset A(V)$:

Repeating the nested argument with V_n instead of V_1, we have that $\overline{A(V_{n+1})}$ has a non-empty interier, so if $y_1 \in \overline{A(V_1)}$, then given $y_n \in \overline{A(V_n)}$, we see

$$(y_n - A(V_{n+1})) \cap A(V_n) \neq \emptyset,$$

and

$$(y_n - \overline{A(V_{n+1})}) \cap A(V_n) \neq \emptyset.$$

Thus, there exists $x_n \in V_n$ with

$$Ax_n \in y_n - \overline{A(V_{n+1})}.$$

We can choose

$$y_{n+1} = y_n - Ax_n,$$

so

$$y_{n+1} \in \overline{A(V_{n+1})}.$$

From $d(x_n, 0) < 2^{-n}r$, partial sums $\sum_{j=1}^{n} x_j$ form a Cauchy sequence:

$$d(\sum_{j=1}^{n} x_j, \sum_{j=1}^{m} x_j) = d(\sum_{j=n}^{m} x_j, 0) < \sum_{j=n}^{m} 2^{-j}r \to 0,$$

which converges by completeness to some $x \in X$.

Next,

$$\sum_{n=1}^{m} Ax_n = \sum_{n=1}^{m}(y_n - y_{n+1}) = y_1 - y_{m+1} \to y_1.$$

So we have

$$y_1 = Ax \in A(V).$$

Finally we want to show that Y is a F-space.