Functional Analysis, Math 7320
Lecture Notes from November 1, 2016

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3.0.10 Corollary. *Let Γ be the set of linear maps from an F-space X to a topological vector space Y and let $B_x = \{Ax : A \in \Gamma\}$ be bounded for all $x \in X$. Then Γ is equicontinuous and uniformly bounded.*

Proof. X is of the second category since it is an F-space, and $X = \{x \in X : B_x \text{ is bounded}\}$. Therefore, by the Banach-Steinhaus theorem, it is the case that Γ is equicontinuous and thus uniformly bounded. □

We specialize to normed spaces.

3.0.11 Theorem (Banach-Steinhaus). *Let Γ be the set of continuous, linear maps from a Banach space X to a normed space Y. If $\sup_{A\in\Gamma} \|Ax\|_Y < \infty$ for all $x \in X$, then there is an $M \geq 0$ such that $\|Ax\|_Y \leq M$ for all $A \in \Gamma$ and all $x \in X$ with $\|x\|_X \leq 1$.*

Proof. Let $X_n = \{x \in X : \sup_{A\in\Gamma} \|Ax\|_Y \leq n\}$. Then $X = \bigcup_{n\in\mathbb{N}} X_n$. Since X is a nonempty, complete metric space, by the Baire category theorem, there is an $m \in \mathbb{N}$ such that X_m has nonempty interior, i.e., there is an $x_0 \in X_m$ and an $\varepsilon > 0$ such that

$$\overline{B_\varepsilon(x_0)} = \{x \in X : \|x - x_0\|_X \leq \varepsilon\} \subseteq X_m.$$

Let $A \in \Gamma$ and let $y \in X$ with $\|y\|_X \leq 1$. Then

$$\begin{aligned}\|Ay\|_Y &= \frac{1}{\varepsilon} \|A(x_0 + \varepsilon y) - Ax_0\|_Y \\ &\leq \frac{1}{\varepsilon} \left(\|A(x_0 + \varepsilon y)\|_Y + \|Ax_0\|_Y\right) \\ &\leq \frac{1}{\varepsilon}(m + m).\end{aligned}$$

Letting $M = 2\varepsilon^{-1}m$ yields the desired result. □

We investigate consequences for sequences of operators.

3.0.12 Corollary. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of continuous, linear maps from an F-space X to a topological vector space Y. If $\lim_{n\to\infty} A_n x = Ax$ exists for all $x \in X$, then A is linear and bounded.*

Proof. Let $x \in X$. Then $\{A_n x\}$ is bounded since $(A_n x)_{n\in\mathbb{N}}$ converges, which implies that $(A_n)_{n\in\mathbb{N}}$ is equicontinuous, i.e., if $W \in \mathcal{U}^Y$, then there is a $V \in \mathcal{U}^X$ such that $A_n V \subseteq W$ for each n, which implies that $\{y \in Y : y = \lim_{n\to\infty} A_n x \text{ for some } x \in V\} \subseteq \overline{W}$, which implies that $AV \subseteq \overline{W}$, which implies that A is bounded and continuous. □

3.0.13 Example. Define $A_n : c_{00} \to c_{00}$ by

$$(A_n x)_m = \begin{cases} m x_m & m \leq n, \\ 0 & m > n, \end{cases}$$

where c_{00} is the set of all sequences that are eventually zero. Then $\|A_n\|_{\mathsf{op}} = n < \infty$, which implies that each A_n is bounded. Moreover, for each $x \in c_{00}$, it is the case that $Ax = \lim_{n\to\infty} A_n x$ exists since $\lim_{n\to\infty} A_n x = (x_1, 2x_2, \dots, m x_m, 0, 0, \dots) \in c_{00}$. However, letting $e_m = (0, 0, \dots, 0, 1, 0, 0, \dots)$, where the 1 is at the m-th position, yields that

$$\sup_{\substack{x \in c_{00} \\ \|x\|_\infty \leq 1}} \|Ax\|_\infty \geq \sup_{m\in\mathbb{N}} \|Ae_m\|_\infty = \sup_{m\in\mathbb{N}} m = \infty.$$

This does not contradict uniform boundedness since c_{00} is not complete: to see that c_{00} is not complete, let

$$a_n = \left(1, \frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{n}}, 0, 0, \dots\right) \in c_{00}.$$

Then $(a_n)_{n\in\mathbb{N}}$ is Cauchy since for every $\varepsilon > 0$, choosing $N \in \mathbb{N}$ such that $1/\sqrt{N} < \varepsilon$ yields that

$$\|a_i - a_j\|_\infty = \frac{1}{\sqrt{j+1}} < \varepsilon$$

whenever $i, j \geq N$, where, without loss of generality, $i > j$. However, $\lim_{n\to\infty} a_n \notin c_{00}$. Moreover, enlarging c_{00} to ℓ^∞ yields that $a = \lim_{n\to\infty} a_n \in \ell^\infty$ but $Aa \notin \ell^\infty$ since $(Aa)_m = \sqrt{m}$, which implies that $\|Aa\|_\infty = \infty$, i.e., we lose point-wise convergence.

3.1 The Open Mapping Theorem

If X and Y are Hausdorff topological spaces, X is compact, and $f : X \to Y$ is continuous and surjective, then f is open.

Due to compactness, this implies that we get open maps on locally-compact topological vector spaces, which are finite dimensional and thus uninteresting for our purposes.

Therefore, we derive an analogous result.

3.1.14 Theorem. *If $A : X \to Y$ is a continuous, linear map from an F-space X to a topological vector space Y and AX is of the second category, then $AX = Y$, A is open, and Y is an F-space.*

Proof. Assume that A is open and let $V \in \mathcal{U}^X$ be balanced. Then there is a balanced $W \in \mathcal{U}^Y$ such that $AX \supseteq AV \supseteq W$. By linearity of A, it is the case that $AX \supseteq A(nV) \supseteq nW$ for all $n \in \mathbb{N}$. Therefore,

$$AX \supseteq \bigcup_{n=1}^{\infty} (nW) = Y.$$

Moreover, we have that $\overline{nW} \supseteq (nW)^{\circ} = nW^{\circ} \neq \varnothing$, which implies that Y is of the second category.

To see that A is open, let d be an invariant metric on X that induces the topology of X, let $V_0 = B_r(0) \subseteq V$, and define $V_n = B_{2^{-n}r}(0)$. By the triangle inequality, $V_1 \supseteq V_2 - V_2$, which implies that $\overline{AV_1} \supseteq \overline{AV_2 - AV_2} \supseteq \overline{AV_2} - \overline{AV_2}$. Therefore,

$$AX = A\left(\bigcup_{k=1}^{\infty} (kV_2)\right) = \bigcup_{k=1}^{\infty} (kAV_2),$$

which implies that one of the kAV_2 is of the second category. Since scaling by k is a homeomorphism, it is the case that AV_2 is of the second category, which implies that $\overline{AV_2}$ has nonempty interior. Repeating the nested argument with V_n instead of V_1 yields that $\overline{AV_{n+1}}$ has nonempty interior. Let $y_1 \in \overline{AV_1}$. Then $y_n \in \overline{AV_n}$ yields that $\left(y_n - \overline{AV_{n+1}}\right) \cap AV_n \neq \varnothing$ and $(y_n - AV_{n+1}) \cap AV_n \neq \varnothing$. So there is an $x_n \in V_n$ such that $Ax_n \in \left(y_n - \overline{AV_{n+1}}\right)$. Let $y_{n+1} = y_n - Ax_n$ so that $y_n \in \overline{AV_{n+1}}$. From $d(x_n, 0) < 2^{-n}r$, we see that the partial sums $\sum_{k=1}^{n} x_k$ form a Cauchy sequence, which converges by completeness:

$$\sum_{n=1}^{m} Ax_n = \sum_{n=1}^{m} (y_n - y_{n+1}) = y_1 - y_{m+1} \xrightarrow{m\to\infty} 0.$$

Hence, $y_1 = Ax \in AV$. □