Functional Analysis, Math 7320
Lecture Notes from November 01, 2016

taken by Qianfan Bai

Last Time

- Equicontinuity and uniform boundedness
- Baire category and uniform boundedness in topological vector space

3.1.0 Corollary. *Let Γ be a collection of continuous linear maps from an F-space X into a topological vector space Y, and if the set*

$$B_x = \{Ax : A \in \Gamma\}$$

is bounded in Y for every $x \in X$, then Γ is equicontinuous and hence uniformly bounded.

Proof. This is an immediate consequence of the theorem given in previous notes, because an F-space is of the second category. Here, we have that

$$B := \big\{x \in X : \{Ax : A \in \Gamma\} \text{ is bounded}\big\},$$

i.e. $B := \{x \in X : B_x \text{ is bounded}\} = X$ is of second category, so the preceding theorem applies. □

We specialize further to X, Y normed spaces.

3.1.1 Theorem. *(Banach-Steinhaus) If X is a Banach space and Y is a normed space, and Γ is a family of continuous linear maps from X to Y, then if for each $x \in X$,*

$$\sup_{A\in\Gamma} \|Ax\| < \infty,$$

then $\sup_{A\in\Gamma} \|A\| < \infty$.

Proof. We know that X is of the second category, so from the previous theorem Γ is equicontinuous, which implies that it is uniformly bounded. That is there exists an $M \geq 0$ such that the (bounded) unit ball in X is mapped into a bounded set in Y,

$$\|Ax\| \leq M, \qquad (\forall x \in X, \quad \|x\| \leq 1)$$

So $\|Ax\| \leq M\|x\|$, hence $\sup_{A\in\Gamma} \|A\| \leq M < \infty$ □

Next, we investigate consequences for sequences of operators.
Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of continuous linear maps from an F-$space$ to a topological vector space Y, and for each $x \in X$, $\lim_{n\to\infty} A_n(x) = A(x)$ exists.

3.1.2 Question. Is A bounded or continuous?
Banach Steinhaus theorem implies the equicontinuity of $(A_n)_{n\in\mathbb{N}}$, hence A is bounded and continuous.

3.1.3 Corollary. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of continuous linear maps from an F-$space$ to a topological vector space Y, and suppose for all $x \in X$, $A(x) = \lim_{n\to\infty} A_n(x)$ exists, then A is bounded and continuous.*

Proof. Let $x \in X$. By the convergence of $(A_n x)_{n\in\mathbb{N}}$ in Y, $\{A_n x\}_{n\in\mathbb{N}}$ is bounded in Y, so $(A_n)_{n\in\mathbb{N}}$ is equicontinuous by Banach-Steinhaus for F-spaces.
Thus, if W is a neighborhood of 0 in Y, then there is some neighborhood V of 0 in X with $A_n(V) \subset W$ for all $n \in \mathbb{N}$, then

$$\{y \in Y : y = \lim_{n\to\infty} A_n(x) \text{ for } x \in V\} \subset \overline{W},$$

so $A(V) \subset \overline{W}$. Hence A is bounded and continuous.

□

3.1.4 Example. Consider the vector space

$$c_{00} = \{x \in l^\infty, \text{there is } n \in \mathbb{N} \text{ such that } x_j = 0 \text{ for all } j \geq n\}.$$

Let $A_n := c_{00} \to c_{00}$ be given by

$$(A_n x)_m = (x_1, 2x_2, \dots, mx_m, 0, 0, \dots).$$

That is for $m \leq n$, $(A_n x)_m = mx_m$, and for $m > n$, $(A_n x)_m = 0$.
A_n is bounded and continuous as for $\|x\| \leq 1$, we have

$$\|A_n x\| = \|(x_1, 2x_2, \dots, mx_m, 0, 0, \dots)\| = \sup_{1\leq m\leq n} |mx_m| \leq n\|x\| \leq n < \infty.$$

For any $x \in c_{00}$, there exists $m \in \mathbb{N}$ such that $x = (x_1, x_2, \dots, x_m, 0, 0, \dots)$. Hence for $n \geq m$, $A_n x$ is constant and equal to $(x_1, 2x_2, \dots, mx_m, 0, 0, \dots)$, which proves that $(A_n x)$ converges, i.e. $Ax = \lim_{n\to\infty} A_n x$ exists.
However, (A_n) is not uniformly bounded as we can see looking at the elements

$$e_m = (0, 0, \dots, 0, 1, 0, \dots, 0, \dots)$$

which have all terms equal to 0 except the m-th one which is equal to 1, we have

$$\|e_m\| = 1 \quad and \quad \|A_n e_m\| = \|(0, 0, \dots, 0, m, 0, \dots, 0, \dots)\| = m.$$

However, this does not contradict Banach-Steinhaus Theorem. Because c_{00} is not complete as we can see considering the sequence (y_m) of c_{00} where $y_m = (1, \frac{1}{2}, \dots, \frac{1}{m}, 0, 0, \dots)$. (y_m) is a Cauchy sequence as for $1 \leq p < q : \|y_q - y_p\| = \frac{1}{p+1}$ but (y_m) doesn't converge in c_{00}.

3.1.5 Remark. If we wanted to replace c_{00} with $c_0 := \{(x_j)_{j\in\mathbb{N}} : \lim_{j\to\infty} x_j = 0\}$, then, however, with $x = (\frac{1}{\sqrt{n}})_{n\in\mathbb{N}} \in c_0$, we would have

$$(Ax)_n = \frac{n}{\sqrt{n}} = \sqrt{n} \quad \to \infty,$$

so $Ax \notin c_0$. We loose pointwise convergence.

3.2 The Open Mapping Theorem

Let X, Y be two topological spaces. We say $f : X \to Y$ is open at point $p \in X$ if $f(V)$ contains a neighborhood of $f(p)$ whenever V is a neighborhood of p. We say that f is open if $f(U)$ is open in Y whenever U is open in X. A linear mapping between two topological vector spaces is open iff it is open at the origin.

3.2.6 Lemma. *Let X and Y be Banach spaces and $A \in \mathcal{B}(X,Y)$ is onto, then A is an open mapping.*

Proof. It suffices to show that $A(B_1(0))$ contains $B_\delta(0)$ for some $\delta > 0$. We break its proof into two parts:
First, $\overline{A(B_{\frac{1}{2}}(0))}$ contains $B_\delta(0)$ for some $\delta > 0$. Since A is onto,

$$Y = \bigcup_{k=1}^{\infty} A(B_k(0)).$$

And since Y is complete, there exists k such that $\overline{A(B_k(0))}$ contains an open ball $B_r(y_0)$ for some $y_0 \in Y$ and $r > 0$. Hence,

$$B_r(0) = B_r(y_0) - y_0 \subset \overline{A(B_k(0))} - \overline{A(B_k(0))} \subset \overline{A(B_{2k}(0))}.$$

Hence, $B_\delta(0) \subset \overline{A(B_{\frac{1}{2}}(0))}$ if we take $\delta = \frac{r}{4k}$.
Second, $\overline{A(B_{\frac{1}{2}}(0))} \subset A(B_1(0))$. Fix $y_1 \in \overline{A(B_{\frac{1}{2}}(0))}$. Assume $n \geq 1$ and $y_n \in \overline{A(B_{\frac{1}{2^n}}(0))}$ has been chosen. From the above first step we have that for any $r > 0$, $\overline{A(B_r(0))}$ contains a neighborhood of 0. Hence

$$(y_n - \overline{A(B_{\frac{1}{2^{n+1}}}(0))}) \cap A(B_{\frac{1}{2^n}}(0))$$

is nonempty, and we can choose $x_n \in B_{\frac{1}{2^n}}(0)$ and $y_{n+1} \in \overline{A(B_{\frac{1}{2^{n+1}}}(0))}$ such that

$$y_n - y_{n+1} = Ax_n \quad .$$

Sum over n, we have

$$y_1 - y_{n+1} = A(\sum_{k=1}^{n} x_k) \quad \to Ax$$

where $x = \sum_{k=1}^{\infty} x_k \in B_1(0)$ is well defined since $\|x_k\| < \frac{1}{k}$. On the other hand, the continuity of A implies $y_n \to 0$ as $n \to \infty$, so we have $y_1 = Ax \in A(B_1(0)$.

□

Now we derive an analogous result for maps between F-spaces.

3.2.7 Theorem. *Let $A : X \to Y$ be a map from an F-space X to a topological vector space Y which is continuous and linear, and $A(X)$ is of the second category in Y, then $A(X) = Y$, A is open mapping and Y is an F-space.*

Proof. To prove that A is open mapping, we only need to show it is open at 0, i.e. an open neighborhood of 0 in X is mapped to an open neighborhood of 0 in Y. After proving this, we have for any balanced $V \in \mathcal{U}^X$, there is a balanced $W \in \mathcal{U}^Y$ such that

$$W \subset A(V) \subset A(X),$$

but by linearity, for any $n \in \mathbb{N}$,

$$nW \subset A(nV) \subset A(X).$$

Taking the union over all $n \in \mathbb{N}$ gives

$$Y = \bigcup_{n \in \mathbb{N}} nW \subset A(X).$$

Also, we see that $\emptyset \neq nW^0 = (nW)^0 \subset \overline{nW}$, so Y is of the second category. Thus, it remains to show that A is open at 0.
Let d be an invariant metric on X that is compatible with the topology of X and let $V_0 = B_r(0) \subset V$ and define $V_n = B_{2^{-n}r}(0)$. We will prove there is $W \in \mathcal{U}^Y$ such that

$$W \subset \overline{A(V_1)} \subset A(V).$$

From the triangle inequality, we get $V_1 \supset V_2 - V_2$, so

$$\overline{A(V_1)} \supset \overline{A(V_2) - A(V_2)} \supset \overline{A(V_2)} - \overline{A(V_2)}.$$

Next, we want to show $\overline{A(V_2)}$ has non-empty interior to get $W \in \mathcal{U}^Y$ with $W \subset \overline{A(V_1)}$.
Next, $A(X) = A(\bigcup_{k=1}^{\infty} kV_2) = \bigcup_{k=1}^{\infty}(kA(V_2))$ because V_2 is a neighborhood of 0. At least one $kA(V_2)$ is therefore of the second category in Y. Since $y \to ky$ is a homeomorphism of Y onto Y, $A(V_2)$ is of the second category in Y. Its closure therefore has nonempty interior.
We still need to show $\overline{A(V_1)} \subset A(V)$.
Repeating the nested argument with V_n instead of V_1, we have that $\overline{A(V_{n+1})}$ has a non-empty interier, so if $y_1 \in \overline{A(V_1)}$, then given $y_n \in \overline{A(V_n)}$, we see

$$(y_n - A(V_{n+1})) \cap A(V_n) \neq \emptyset$$

and

$$(y_n - \overline{A(V_{n+1})}) \cap A(V_n) \neq \emptyset.$$

Thus, there exists $x_n \in V_n$ with

$$Ax_n \in y_n - \overline{A(V_{n+1})}$$

We can choose

$$y_{n+1} = y_n - Ax_n$$

so

$$y_{n+1} \in \overline{A(V_{n+1})}.$$

From $d(x_n, 0) < 2^{-n}r$, partial sums $\sum_{j=1}^{n} x_j$ form a Cauchy sequence, which converges by completeness to some $x \in X$.
Next, $\sum_{n=1}^{m} Ax_n = \sum_{n=1}^{m}(y_n - y_{n+1}) = y_1 - y_{m+1} \to 0$ as $m \to \infty$.
So we have $y_1 = Ax \in A(V)$. This gives $\overline{A(V_1)} \subset A(V)$. Hence A is an open mapping.
Finally, we want to show that Y is an F-$space$. (See next lecture notes).

□