Functional Analysis, Math 7320
Lecture Notes from November 8, 2016

taken by Worawit Tepsan

4 Convexity

Warm-Up: Internal Points vs Convexity

Look briefly on the definitions of internal points and convexity, they are similar. One natural question to ask is that "does an element in a convex set is an internal point of the set?" This is not true in general. We provide a counter example that shows a non-empty convex set does not need to have an internal point. Recall that c_{00} is the set of all sequences of real numbers with a finite non-zero terms. Define c_{0+} to be a subset of c_{00} which its last non-zero entry is positive, i.e.,

$$c_{0+} = \{(x_n)_{n\in\mathbb{N}} \in c_{00} : x_{\max\{n : x_n \neq 0\}} > 0\}.$$

First, we show that c_{0+} is convex. Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_m)_{m\in\mathbb{N}} \in c_{0+}$ be elements in c_{0+}. Let x_n and y_m be the last non-zero terms of x and y respectively. Thus, x_n and y_m are positive. Let $t \in [0,1]$. If $m = n$, the last non zero term of $tx + (1-t)y$ is $tx_n + (1-t)y_m > 0$. Thus, $tx + (1-t)y \in c_{0+}$. For $m \neq n$, with out loss of generality, we assume $m > n$. If $t = 0$, $tx + (1-t)y = y$ which is in c_{0+}. Similarly for $t = 1$, $tx + (1-t)y = x \in c_{0+}$. If $t \in (0,1)$, then the last term of $tx + (1-t)y$ is $(1-t)y_m > 0$. In conclusion, $tx + (1-t)y \in c_{0+}$ for all $x, y \in c_{0+}$ and $t \in [0,1]$. Therefore, c_{0+} is convex.

4.0.1 Question. Does c_{0+} have internal points?

4.0.2 Answer. The set c_{0+} has no internal point because if $x = (x_n)_{n\in\mathbb{N}} \in c_{0+}$ such that $x_n > 0$ is the last non-zero term. We can pick $y = (y_n)_{n\in\mathbb{N}} \in c_{0+}$ such that $y_{n+1} > 0$ is the last non-zero entry. Then, the last term of $x - \varepsilon y \notin c_{0+}$ is $-\varepsilon y_{n+1} < 0$ for any $\varepsilon > 0$. This shows any x in c_{0+} can not be an internal point, i.e., c_{0+} has no internal point.

4.1 Hahn Banach Theorem

For a topological vector space X, a **functional** on X is a function f that maps X to $\mathbb{R}$ or $\mathbb{C}$. If f maps to $\mathbb{R}$, f is called a **real functional**, and it will be called a **complex functional** if it maps to $\mathbb{C}$. Assume that we have a real linear functional f defined on a subspace of the whole space. Under some constrains of f, we might be able to extend f to a functional on X which preserves a property of the original functional. Last time, we introduce the statement of Hahn Banach Theorem which states that a linear functional on a subspace, which is dominated by a proper functional on the whole space, can be extended to a linear functional on the whole space which is still dominated with the same functional. In this lecture, we prove the Hahn Banach Theorem.

4.1.3 Theorem (Hahn Banach Theorem on $\mathbb{R}$). *Let V be a real vector space and $p : V \to \mathbb{R}$ satisfying*

- $p(x+y) \leq p(x) + p(y)$ *for all $x, y \in V$, and*
- $p(\alpha x) = \alpha p(x)$ *for all $x \in V$ and $\alpha \in \mathbb{R}^+$.*

Let $Y \subseteq V$ be a linear subspace and a linear functional $f : X \to \mathbb{R}$ such that $f \leq p|_Y$. Then, there is a linear functional $F : V \to \mathbb{R}$ such that $F|_Y = f$ and also $F \leq p$.

Proof. Consider the collection of linear extensions of f, (g, V_g) where $Y \subseteq V_g \subseteq V$, $g|_Y = f$, and $g \leq p|_{V_g}$. We define a partial order base on set inclusion, i.e., $(g, V_g) \leq (h, V_h)$ if $V_g \subseteq V_h$ and $h|_{V_g} = g$. Next, let $\{(g, V_g)\}_{g \in G}$ be a chain of the partial order and set $D = \bigcup_{g \in G} V_g$, then F defined on D by $F(x) = g(x)$ if $x \in V_g$. Then F satisfies $F|_D \leq p|_D$. Hence, F is an upper bound for the chain. By Zorn's Lemma, there exists a maximal element (F, V_F) among all linear extensions of f dominated by p. It remains to show that $V = V_F$. If this is not the case, take $x_0 \in V \setminus V_F$ and $V_1 = \text{span}\{V_F, x_0\}$. Each vector $x \in V_1$ can be written uniquely as $x = y + \alpha x_0$ where $y \in V_F$ and $\alpha \in \mathbb{R}$. Next, let $f_1 : V_1 \to \mathbb{R}$ be given by $f_1(x_0) = \beta$. So,

$$f_1(y + \alpha x_0) = f_1(y) + \alpha f_1(x_0) = f_1(y) + \alpha\beta.$$

Since we want to have f_1 as an extension of F, $f_1(y) = F(y)$. In addition, we need f_1 to be dominated by p. Therefore, we will choose a proper $\beta \in \mathbb{R}$ so that $f_1 \leq p|_{V_1}$. As explain before, we want

$$f_1(y + \alpha x_0) = f_1(y) + \alpha\beta \leq p(y + \alpha x_0).$$

In particular, for $\alpha = \pm 1, y, y' \in V_F$, we need

$$f_1(y) + \beta \leq p(y + x_0),$$

and

$$f_1(y') - \beta \leq p(y' - x_0).$$

We combine two inequalities, we obtain

$$f_1(y') - \beta - p(y' - x_0) \leq 0 \leq p(y + x_0) - (f_1(y) + \beta).$$

By adding β to both sides,

$$f_1(y') - p(y' - x_0) \leq p(y + x_0) - f_1(y).$$

This inequality is satisfied because

$$\begin{aligned} f_1(y') + f_1(y) &= f_1(y' + y) = f_1(y' + x_0 + y - x_0) \\ &\leq F(y + x_0 + y' - x_0) \leq p(y + x_0 + y' - x_0) \\ &\leq p(y + x_0) + p(y' - x_0). \end{aligned}$$

By rearranging the terms, we have the earlier inequality. This implies

$$a = \sup_{y' \in V_F} (f_1(y') - p(y' - x)) \leq \inf_{y \in V_F} (p(y + x_0) - f_1(y)) = b.$$

Choose $\beta \in [a, b]$,

$$\sup_{y' \in V_F} (f_1(y') - \beta - p(y' - x)) \leq 0 \leq \inf_{y \in V_F} (p(y + x_0) - (f_1(y) + \beta)).$$

Consequently, for such a choice of β and $\alpha > 0$,

$$f_1(y + \alpha x_0) = \alpha f_1(y/\alpha + x_0) \leq \alpha p(y/\alpha) + x_0) = p(y + \alpha x_0).$$

If $\alpha = 0$, we get $f_1(y) = F(y) \leq p(y)$. If $\alpha < 0$, we obtain $f_1(y + \alpha x_0) = |\alpha| f_1(y/|\alpha| - x_0) \leq |\alpha| p(y/|\alpha| - x_0) = p(y + \alpha x_0)$. Hence, $f_1|_Y = F$ and $f_1 \leq p|_{V_1}$. So, f_1 is a non-trivial extension of F which contradicts the maximality. □

4.1.4 Remarks. (1) The functional $p : V \to \mathbb{R}$ in the theorem is called a **sublinear functional**. By the properties of p, $p(tx + (1-t)y) \leq p(tx) + (1-t)p(y) = tp(x) + (1-t)p(y)$ for all $x, y \in V$ and $t \in [0, 1]$. Thus, a sublinear functional is convex. Also, $p(0) = p(2 \cdot 0) = 2p(0)$. This implies $p(0) = 0$. Of course, a linear function is sublinear. Also, seminorms and norms are examples of sublinear functional which are not linear. Let K be symmetric convex subset of X. Then, the **Minkowsk function**(Check ref [2]) defined by

$$p(x) = \inf\{\lambda > 0 : x \in \lambda K\}$$

is also a sublinear functional.

(2) As we can see from the proof, in the case when V is a finite dimensional vector space, we can explicitly construct an extension of f and it can be done in a finite steps. However, in the case when V is an infinite dimensional vector space, the extension might not be able to be done by a finite steps. Thus, we need the method of transfinite induction and apply the Zorn's lemma to get the result. Therefore, we might not be able to see explicitly what is going to be an extension of f on V. However, by the theorem, we know the existence of such an extension.

(3) From how the proof extends a linear functional, if $a \neq b$, there are infinitely many ways we can choose $\beta \in [a, b]$. Therefore,there exist abundant of linear functional on X that extend a linear functional on Y. Thus, we have existence of such a functional but no uniqueness.

The Hahn-Banach theorem can be extended to a linear functional on a complex vector space. Before we see the theorem, we note some facts for a linear functional on a complex vector space.

4.1.5 Facts. (1) Let $(X, +, \times)$ be a complex vector space. We can can consider X as a real vector space by the operations $+_{\mathbb{R}}$ and $\times_{\mathbb{R}}$ defined by $x +_{\mathbb{R}} y = x + y$ and $\alpha \times_{\mathbb{R}} x = \alpha \times x$ for $x, y \in X$ and $\alpha \in \mathbb{R}$ (For instance, $\mathbb{C}$ is a one dimensional complex vector space. However, we can consider $\mathbb{C} = \mathbb{R}^2$ as a two dimensional real vector space).

(2) For a linear functional f on X, we write $f(x) = u(x) + iv(x)$ where $u : X \to \mathbb{R}$ is the real part of f and $v : X \to \mathbb{R}$ is the imaginary part of f. By replacing x by ix, $f(ix) = u(ix) + iv(ix)$. By linearity of f, we have $if(x) = u(ix) + iv(ix)$. Thus, $f(x) = v(ix) - iu(ix)$. Comparing the imaginary part of f, we obtain $v(x) = -u(ix)$. Therefore, we can write $f(x) = u(x) - iu(ix)$.

As we have seen from the facts above, we can write a linear functional on X as $f(x) = u(x) - iu(ix)$. The following lemma shows that $u(x)$ is actually a real linear functional. Conversely, if we have a real linear functional u on X, we can define a complex linear functional which $u(x)$ is a real part.

4.1.6 Lemma. *Let X be a complex vector space and f be a complex linear functional on X. Then, $Re(f)$ is a real linear functional on X. Conversely, if u is a real linear function on X, $f(x) = u(x) - iu(ix)$ is a complex linear functional.*

Proof. Let f be a complex linear functional on X and $u = Re(f)$. Note that, we can write the real part of f as the average of f and its conjugate, i.e.,

$$u(x) = \frac{f(x) + \overline{f(x)}}{2}.$$

Because of the linearity of f combining with the properties of conjugate and then arranging terms,

$$\begin{aligned} u(x+y) &= \frac{f(x+y) + \overline{f(x+y)}}{2} = \frac{f(x) + f(y) + \overline{f(x) + f(y)}}{2} \\ &= \frac{f(x) + f(y) + \overline{f(x)} + \overline{f(y)}}{2} = \frac{f(x) + \overline{f(x)}}{2} + \frac{f(y) + \overline{f(y)}}{2} \\ &= u(x) + u(y), \end{aligned}$$

and for $\alpha \in \mathbb{R}$,

$$u(\alpha x) = \frac{f(\alpha x) + \overline{f(\alpha x)}}{2} = \frac{\alpha f(x) + \overline{\alpha f(x)}}{2} = \frac{\alpha f(x) + \alpha \overline{f(x)}}{2} = \frac{\alpha(f(x) + \overline{f(x)})}{2} = \alpha u(x).$$

Thus, the real part of f is a real linear functional on X.

Conversely, let $u(x)$ be a real linear functional on X, $x, y \in X$ and $\alpha = a + bi \in \mathbb{C}$. Then, by the linearity of u,

$$\begin{aligned} f(x+y) &= u(x+y) - iu(i(x+y)) = u(x) + u(y) - i(u(ix) + u(iy)) \\ &= \big(u(x) - iu(ix)\big) + \big(u(y) - iu(iy)\big) = f(x) + f(y). \end{aligned}$$

In addition,

$$\begin{aligned} f(\alpha x) &= u((a+bi)x) - iu(i(a+bi)x) = u(ax + ibx) - iu(iax - bx) \\ &= au(x) + bu(ix) - iau(ix) - bu(x) = (a+bi)(u(x) - iu(ix)) \\ &= \alpha f(x). \end{aligned}$$

Thus, $f(x) = u(x) - iu(ix)$ is a complex linear functional on X.

□

Now, we are going to apply the lemma above to prove the Hahn-Banach theorem for the complex case. The idea of the proof is that we extend the real part of a linear functional by Hahn-Banach theorem and then construct the complex linear functional from the extended functional of the real part.

4.1.7 Theorem. *Let X be a complex vector space and p a seminorm on X. If $Y \subseteq X$ is a subspace, $f : Y \to \mathbb{C}$ be a linear functional satisfying $|f| \leq p|_Y$, then there is a linear functional $F : X \to \mathbb{C}$ with $|F| \leq p$ and $F|_Y = f$.*

Proof. Let u be a real part of f. From the lemma, u is a real linear functional on Y. Then

$$u(x) \leq |u(x) - iu(ix)| = |f(x)| \leq p(x).$$

Therefore, u is satisfied the condition on the Hahn-Banach theorem. Hence, we can extend u to be $U(x)$ on X such that $U(x) \leq p(x)$. Define $F(x) = U(x) - iU(ix)$. Then F is a complex linear functional on X. Also, if $x \in Y$, since U is an extension of u, $F(x) = U(x) - iU(ix) = u(x) - iu(ix) = f(x)$, i.e., F is an extension of f. Let $x \in X$. Choose $\alpha = |Ax|/Ax$ if $Ax \neq 0$ and $\alpha = 0$ if $Ax = 0$. Then, we have $A(\alpha x) = \alpha Ax = |Ax|$. Thus, since $A(\alpha x)$ is real, the imaginary part of $A(\alpha x)$ is 0. Therefore,

$$|Ax| = A(\alpha x) = U(\alpha x) - iA(i\alpha x) = U(\alpha x) \leq p(\alpha x) = |\alpha| p(x) = |(|Ax|/Ax)| p(x) = p(x).$$

This show that F is an extension of f which $|A(x)| \leq p(x)$ for all $x \in X$. $\square$

One consequence from the above theorem is in a normed vector space.

4.1.8 Corollary. *If X is a normed space, $x_0 \in X_1$, then there is a linear functional f such that $f(x_0) = \|x_0\|$ and $|f(x)| \leq \|x\|$ for all $x \in X$.*

Proof. If $x_0 = 0$, let $f = 0$. Otherwise, take $p(x) = \|x\|$, $Y = \text{span}\{x_0\}$ and

$$f(\alpha x_0) = \alpha \|x_0\|.$$

We see that f is a complex linear functional on Y. Also, $|f(x)| = \|x\| \leq p(x)$ for all $x \in Y$. Then, applying the preceding theorem to obtain a linear functional on X that extends f. $\square$

Next, we discuss separation of sets by a linear functional. The question is that "can we find a linear function which the value of f on a set M is always bigger or equal to the value of f on a set N?". So, for given sets M and N, we are searching for a linear functional on X which $f(x) \leq f(y)$ for all $x \in M$ and $y \in N$.

4.1.9 Definition. Let V be a vector space, $M, N \subseteq V$. A linear functional f on V is said to separate M and N if

$$\sup Re(f(N)) \leq \inf Re(f(M)).$$

Note that for a set $S \subseteq V$, $f(S) = \{f(y) : y \in S\}$.

4.1.10 Examples. We provide some obvious examples.

(1) Let $M, N \subseteq \mathbb{R}$. The identity map $Id : \mathbb{R} \to \mathbb{R}$ separates M and N if $\sup\{x \in M\} \leq \inf\{x \in N\}$.

(2) Let $A : \mathbb{R}^2 \to \mathbb{R}$ be a linear functional. Then A can be represented as a vector $A = [a, b]$ and $A(x, y) = ax + by$. Then the normal line $\{(x, y) : ax + by = 0\}$ divides $\mathbb{R}^2$ into two sides. If M is a subset of one side and N is subset of the other, A separates M and N.

(3) In a normed vector space X, let $M = \{x \in X : \|x\| < 1\}$ and $N = \{x \in X : \|x\| > 1\}$. Assume that a non-zero linear functional f separates M and N. Then, without loss of generality assume that $f(x) \leq f(y)$ for all $x \in X$ and $y \in Y$. We can choose a non-zero vector $x \in M$ such that $f(x) > 0$. Then, $y = -2x/\|x\| \in N$ because $\|y\| = 2 > 1$. By linearity of f, $f(y) = -2f(x)/\|x\| < 0 < f(x)$. This contradicts our assumption. Therefore, there is no linear functional separating M and N.

References

[1] Rudin, Walter., *Functional Analysis, 2nd*, McGraw Hill Education, 1973.

[2] https://en.wikipedia.org/wiki/Minkowski_functional.