Functional Analysis, Math 7320
Lecture Notes from November 10, 2016

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3.3.23 Proposition. *A linear functional f separates between 2 sets M and N if and only if it separates between $M - N$ and $\{0\}$.*

Proof. Let V be a vector space over $\mathbb{K}$, $M, N \subseteq V$ two subsets, and $f : V \to \mathbb{K}$ a linear functional. Then $x \mapsto \text{Re}f(x)$ is a real-linear[1] functional $V \to \mathbb{R}$, when V is considered as an $\mathbb{R}$-vector space. Hence, for each $(x, y) \in M \times N$,

$$\text{Re}f(x) \leq \text{Re}f(y) \Longleftrightarrow \text{Re}f(x - y) \leq 0 = \text{Re}f(0).$$

The result follows by definition. □

Warm-up: Recall that C_{0+} is a convex subset of C_{00}.

3.3.24 Claim. *There is no linear functional on C_{00} that separates[2] C_{0+} from $\{0\}$.*

Proof. Suppose for contradiction that there is a non-zero functional $f : C_{00} \to \mathbb{R}$ such that $f(C_{0+}) \subseteq \mathbb{R}^+$. Call e_j $(j \in \mathbb{N})$ the canonical vector from C_{00} which has its j^{th} entry equal to 1 and all the other ones to 0. Clearly $e_j \in C_{0+}$ and $f(e_j) \geq 0$ for all $j \in \mathbb{N}$. Furthermore, the fact that $\{e_j\}_{j \in \mathbb{N}}$ forms a basis for C_{00} and the assumption that $f \neq 0$ imply that $f(e_j) > 0$ for some $j \in \mathbb{N}$. If $f(e_{j+1}) = 0$, then $-e_j + e_{j+1} \in C_{0+}$, and $f(-e_j + e_{j+1}) < 0$ generates a contradiction. If $f(e_{j+1}) > 0$, we define

$$v := -e_j + \frac{f(e_j)}{2f(e_{j+1})} e_{j+1} \in C_{0+},$$

which yields, using linearity of f, $f(v) < 0$, and also generates a contradiction. □

3.3.25 Question. Is it possible to separate $\{0\}$ from any other set in C_{00} with a linear functional?

3.3.26 Theorem (Masur)**.** *Let M, N be disjoint non-empty convex sets in a vector space V. If at least one of these sets, say M, has an internal point, then there exists a non-zero linear functional that separates M and N.*

We will only consider the real case here, that is when $\mathbb{K} = \mathbb{R}$. We will first present a lemma.

[1] Meaning that the scalar field is $\mathbb{R}$.

[2] The fact that $0 \in C_{0+}$ does not impede separation by itself, as we defined separation with a large inequality.

3.3.27 Lemma. *1. A linear functional $f : V \to \mathbb{R}$ separates M and N if and only if it separates $M - p$ and $N - p$.*

2. A point $p \in M$ is an internal point of M if and only if 0 is an internal point of $M - p$.

3. (a) For any $p \in V$, the set M is convex if and only if $M - p$ is convex.

(b) If $M, N \subseteq V$ are convex, then so is $M - N$.

Proof of lemma. 1. This is a direct consequence of the following equalities,

$$(\sup f(M)) - f(p) = \sup (f(M) - f(p)) = \sup f(M - p),$$

which also hold true for $(\inf f(N)) - f(p)$.

2. This is immediate from the definition, given that for all $x \in V$ and all $\epsilon \in (-1, 1)$, $p + \epsilon x \in M \Leftrightarrow \epsilon x \in M - p$.

3. We start by proving (b). Let $m_1, m_2 \in M$, $n_1, n_2 \in N$ and $0 \leq \lambda \leq 1$. The result is immediate considering,

$$\lambda(m_1 - n_1) + (1 - \lambda)(m_2 - n_2) = \lambda m_1 + (1 - \lambda)m_1 - (\lambda n_1 + (1 - \lambda)n_2) \in M - N.$$

Now, (b) implies one implication in (a), when $N = \{p\}$. For the converse, assume that $M - p$ is convex, let $m_1 - p, m_2 - p \in M - p$ and $\lambda \in [0, 1]$. We obtain the result as follows:

$$\lambda(m_1 - p) + (1 - \lambda)(m_2 - p) \in M - p \Rightarrow \lambda m_1 + (1 - \lambda)m_2 \in M.$$

□

Proof of Masur's theorem in the real case. By lemma **??**, we may assume without loss of generality that 0 is an internal point of M. Next, let $x_0 \in N$ and define $K := M - N + x_0$. The first part of lemma **??** together with proposition **??** imply the following chain of equivalence: A linear functional separates between $\{x_0\}$ and K, if and only if it separates between $\{0\}$ and $M - N$, if and only if it separates between M and N. So our task, at this point, becomes to prove the existence of a non-zero linear functional on V that separates between K and $\{x_0\}$. For this, we will use the Minkowski functional $\mu_K : V \to [0, \infty)$, of K. Recall that,

$$\mu_K(x) := \inf \left\{ t > 0 \ : \ t^{-1}x \in K \right\} \qquad (x \in V)$$

We observe the following facts:

1. Since 0 is an internal point of M, the point $-x_0$ is an internal point of $M - N$ and therefore, 0 *is an internal point of* K.

2. The set K is *convex* by the last part of lemma **??**, and it is *absorbing* since 0 is an internal point of it[3].

[3] See the remark in the notes from 3 November 2016, just after the "warm-up" paragraph.

3. The Minkowski functional μ_K is subadditive and positive homogeneous, the latter meaning that $\mu_K(\alpha x) = \alpha\mu_K(x)$ for all $x \in V$ and $\alpha \geq 0$. This result corresponds to theorem 1.35 in the second edition of the book "Functional Analysis" from W. Rudin.

4. The point x_0 *is not in* K, because if it were, there would exist $(m, n) \in M \times N$ such that $m - n = 0$, which is impossible since $M \cap N = \emptyset$.

5. If a point $x \in V$ satisfies $\mu_K(x) < 1$, then there exists a $t \in (0, 1)$ such that $t^{-1}x \in K$. This implies that $x = tk$ for some $k \in K$. But this in turn implies that $x \in K$ because K is convex and contains 0. So our previous fact implies that $\mu_K(x_0) \geq 1$.

Call span$\{x_0\}$ the linear subspace of V generated by $\{x_0\}$. We may define a *non-zero* linear functional f : span$\{x_0\} \to \mathbb{R}$ by $f(\alpha x_0) := \alpha\mu_K(x_0)$. If $\alpha > 0$, fact 3 tells us that $f(\alpha x_0) \leq \mu_K(\alpha x_0)$, and if $\alpha < 0$, then $f(\alpha x_0) = \alpha\mu_K(x_0) \leq 0 \leq \mu_K(\alpha x_0)$. So, we are exactly in the context of the Hahn-Banach theorem (real version) exposed in the notes from 8 November 2016. Call F the linear extension of f dominated by μ_K. If $x \in K$, then $\mu_K(x) \leq 1$ by definition[4]. On the other hand, since F agrees with f on span$\{x_0\}$, fact 5 yields $F(x_0) = f(x_0) = \mu_K(x_0) \geq 1$, which finishes the proof. □

3.3.28 Exercise. Prove the previous theorem in the complex case.

3.3.29 Corollary. *Let X be a locally convex TVS and K_1, K_2 two disjoint convex sets such that at least one of them has non-empty interior. Then, there exists a non-zero linear functional that separates K_1 and K_2.*

Proof. If K_1 has an interior point $x_0 \in K_1^\circ$, then there exists a convex balanced neighborhood $U \in \mathcal{U}$ such that $x_0 + U \in K_1^\circ$. Furthermore, for any $y \in X$, by continuity of scalar multiplication, there exists an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq U$. Therefore, $x_0 + (-\epsilon, \epsilon)y \subseteq x_0 + U$ and x_0 is an internal point of K_1. The result follows now from an application of the previous separation theorem **??**. □

3.3.30 Corollary. *In a locally convex TVS X, the dual space X^* of continuous linear functionals separates points in X.*

Proof. By the Hausdorff property, given two distinct points $x \neq y$ in X, there is a convex balanced neighborhood $V \in \mathcal{U}$ such that,

$$(x + V) \cap \{y\} = \emptyset,$$

so the preceding corollary applies. □

In applications, it is often desirable to have strict separation, whence the following theorem.

3.3.31 Theorem. *Let V be a vector space and $K \subseteq V$ a convex subset, disjoint from K, whose points are* all *internal. Let D be an affine subspace (i.e. $D = x + W$ for some subspace $W \subseteq V$ and point $x \in V$). Then, there exists a linear functional f such that $f(D) = 0$ and $f(K) \cap \{0\} = \emptyset$.*

[4] Since $1^{-1}x \in K$.

Proof. Up to translating both K and D appropriately, we may assume without loss of generality that D is a linear subspace of V. By Masur's separation theorem **??**, there exist both a linear functional $F : V \to \mathbb{K}$ and $\beta \in \mathbb{R}$ such that,

$$\sup \mathsf{Re} F(K) \leq \beta \leq \inf \mathsf{Re} F(D).$$

By letting $f(x) = \mathsf{Re} F(x)$, we notice that, since $0 \in D$,

$$\beta \leq 0 = f(0) = F(0).$$

To be continued... □