Functional Analysis, Math 7320
Lecture Notes from November 10, 2016

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Last Time

- Hahn Banach over $\mathbb{R}$ and $\mathbb{C}$
- Hahn Banach in spaces with a semi-norm or a norm

Last time, we studied the Hahn-Banach theorem which allows us to extend a linear functional on a subspace to a linear functional on the whole space if that functional is well behaved. Also, we defined the separation of sets. We say sets M and N are separated if there is a linear functional f which $Ref(x) \leq Ref(y)$ for all $x \in M$ and $y \in N$. Next, we are going to apply the Hahn-Banach theorem to find a functional which separates two sets if these two sets are under some constrains.

Convexity

4.1.6 Proposition. *A linear functional f separates between two sets M and N if and only if it separates between $M - N$ and 0.*

Proof. There exists a linear functional f such that for each $x \in M$ and $y \in N$ we have $Ref(x) \leq Ref(y)$ if and only if :

$$Ref(x - y) \leq 0 = Ref(0).$$

□

4.1.7 Examples. We give one example on $\mathbb{R}^2$ that we can separate from 0 and one that we can not separate from 0.

(1) On $\mathbb{R}^2$, a subset of the upper plane is separated from 0. To see this, define $f(x, y) = y$. Then f is a linear fictional and $f(x, y) \geq 0$ for all x in the upper half plane. Therefore, there is a linear functional that separates 0 and any subset of the upper half plane. We can extend this to $\mathbb{R}^n$, we have that any subset of $\{(x_i)_{i=1}^n : x_1 > 0\}$ can be separated from 0.

(2) On $\mathbb{R}^n$, $C = \{x : r_1 < \|x\| < r_2\}$ can not be separated from 0 where $r_2 > r_1 \geq 0$.

Warm up

Recall c_{0+} from last time, a convex subset of c_{00}, there is no linear functional separating c_{0+}

from 0. Let $f \neq 0$ be a linear functional on c_{00} such that $f(c_{0+}) \subset \mathbb{R}^+$. Since the canonical basis vectors are in c_{0+} , so $f(e_j) \geq 0$ for each $j \in \mathbb{N}$. Moreover,the vector basis $\{e_j\}_{j=1}^{\infty}$ spans c_{00} and $f \neq 0$, there is $j \in \mathbb{N}$ such that $f(e_j) > 0$. Now suppose $f(e_{j+1}) = 0$, then we can take $-e_j + e_{j+1} \in c_{0+}$ but $f(-e_j + e_{j+1}) = -f(e_j) < 0$ so this cannot happen, hence $f(e_{j+1}) > 0$ as well. Next, Consider

$$V = -e_j + \frac{f(e_j)}{2f(e_{j+1})} e_{j+1}$$

$$\begin{aligned} f(V) &= -f(e_j) + \frac{f(e_j)}{2f(e_{j+1})} f(e_{j+1}) \\ &= -\frac{1}{2} f(e_j) < 0 \end{aligned}$$

which contradicts that $f(c_{0+}) \subset \mathbb{R}^+$.Hence there is no linear functional separating c_{0+} from origin.

Recall that for a symmetric convex subset K, we define the Minskowski's function of K by

$$\mu_K(x) = \inf\{\alpha > 0 : x \in \alpha K\}.$$

We know that μ_K is a sublinear functional, i.e., for $x, y \in X$ and $\alpha > 0$,

$$\mu_K(x + y) \leq \mu_K(x) + \mu_K(y) \text{ and } \mu_K(\alpha x) = \alpha \mu_K(x).$$

Thus, Minskoski's function satisfies the property of a function p in the Hahn-Banach theorem. Also, if $x \notin K$, $\mu_K(x) \geq 1$. We need these properties of Minskwoki's function to proof the following theorem. To proof the Masur's theorem, first we construct a function on a subspace of V which dominated by a Minskwoki's function. Then, we can extend that linear functional to the whole space.

4.1.8 Theorem. *(Masur's Theorem) Let M, N be disjoint nonempty convex sets in a vector space V. Suppose at leat one of them M or N has an internal point, then there exists a non-zero linear functional that separates M and N.*

Proof. We consider the real case first :
for each point $p \in V$, f separates M and N if and only if it separates $M - p$ and $N - p$. Hence, we can assume 0 is an internal point of M. Then we can take $x_0 \in N$, then $-x_0$ is an initial point of $M - N$, and 0 is an internal point of the convex set $K = x_0 + M - N$ .By the disjointness of M and N, $x_0 \notin K$, if we prove the existence of a linear functional separating K from x_0, then this also separates $M - N$ from 0, hence it separates M and N.
Next, we consider the convex set K with internal point and $x_0 \notin K$:
Let μ_K be the Minkowski functional of K, then by $x_0 \notin K$, $\mu_K(x_0) \geq 1$. On span x_0, we define a linear functional with

$$f(\alpha x_0) = \alpha \mu_K(x_0)$$

so for $\alpha > 0$,

$$f(\alpha x_0) \leq \mu_K(\alpha x_0)$$

and for $\alpha < 0$,

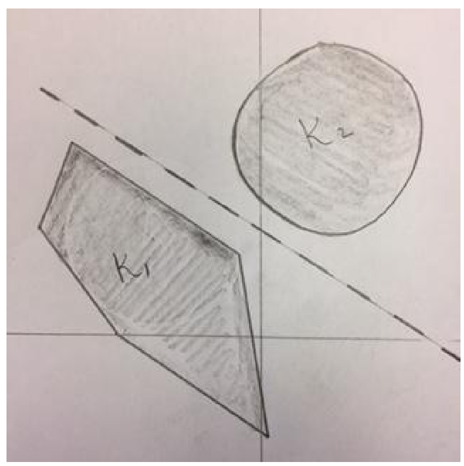
$$f(\alpha x_0) = \alpha\mu_K(x_0) \le 0$$
$$\le \mu_K(\alpha x_0).$$

By Hahn - Banach (over $\mathbb{R}$), there is a linear functional F such that $F|_{span\{x_0\}} = f$ and $F \le \mu_K$. For $x \in K$, $F(x) \le 1$ whereas $F(x_0) = f(x_0) = \mu_K(x_0) \ge 1$,so F separates K from $\{x_0\}$.

□

4.1.9 Remark. For the case when V is a complex vector space, we can consider V as a real vector space. Then, by the previous theorem, there exists a non-zero real linear functional U on V that separates M and N. Define $f(x) = U(x) - iU(ix)$. Then, f will be a complex linear functional on V which real part is U. Therefore, it separates M and N.

The picture below shows that for two disjoint convex set which one of them has internal point, we can find a linear functional separates them as the Masur's theorem.



4.1.10 Corollary. *Let X be a locally convex topological vector space and K_1, K_2 are two disjoint convex sets such that there is at least one has nonempty interior, then there is a non zero linear functional that separates K_1 and K_2.*

Proof. If K_1 has $x_0 \in K_1^0$, then there is a convex $U \in \mathcal{U}$ such that $x_0 + U \subset K_1^0$, and hence for any $y \in X$, by the continuity of $\alpha \mapsto \alpha y$, there is $\epsilon > 0$ such that $(-\epsilon, \epsilon)y \subset U$, so $x_0 + (-\epsilon, \epsilon)y \subset x_0 + U$. Hence, x_0 is a internal point. Now applying Mazur's separation theorem, we obtain the claimed separation. □

4.1.11 Corollary. *In a locally convex topological vector space X, the space of linear functional separates points of X.*

Proof. Since X is locally convex topological vector space and by Hausddorff property, two distinct points x, y have a convex balanced neighborhood V of 0 with $(x+V) \cap \{y\} = \emptyset$, so the preceding corollary applies. □

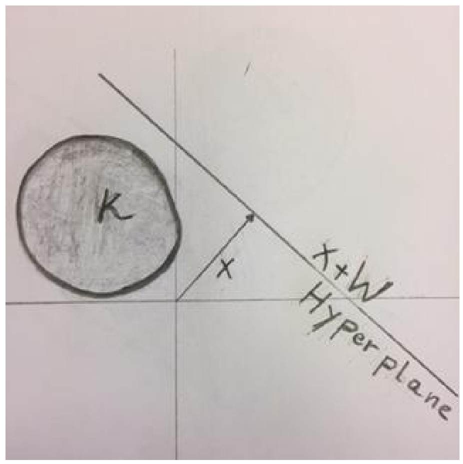
4.1.12 Theorem. *Let V be a vector space and $K \subset V$, a convex susbset whose points are all internal . Let D be an affine subspace(e.g. $x + W$ for some $x \in V$, W a subspace of V) that is disjoint with K, then there is a linear functional such that $f(D) = 0$ and $0 \notin f(K)$.*

Proof. Without loss of generality, we can assume D is a subspace, otherwise we shift both K and D appropriately. By Mazur's separation theorem there is linear functional F and $\beta \in \mathbb{R}$ such that

$$\sup ReF(K) \leq \beta \leq \inf ReF(D)$$

So let $f(x) = ReF(x)$, so if V is over $\mathbb{C}$, then $F(x) = f(x) - if(ix)$. By $0 \in D$, $\beta \leq 0 = f(0) = F(0)$. □

The image below illustrates the theorem.



In application, we would often like to have strict separation, i.e., we want a linear functional F which

$$\sup ReF(K) < \alpha < \inf ReF(D)$$

for some $\alpha \in \mathbb{R}$.