Functional Analysis, Math 7320
Lecture Notes from November 17, 2016

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4 CONVEXITY

4.1 Weak Topology

4.1.1 Theorem. *Let X be a vector space, and X' be a separating vector space of linear functional on X (i.e. for each x, and y in X, $x \neq y$, there exists a linear functional $f : X \to \mathbb{R}$ s.t $f(x) \neq f(y)$)*

Denote τ' for the initial topology induced by X' on X.

Then, (X, τ') is locally convex TVS, and the space of all CTS linear functional is X'.

Proof. cont'd

Last time, we obtain local convex subbase, and $\{0\}$ is closed. What's left to show is vector addition and scalar multiplication are continuous wrt to τ'.

Recall that each subbasis element is of the form:

$$V(f, r) = \{x : |f(x)| < r\}$$

Given $f \in X^*$, we have that:

$$\frac{1}{2}V(f,r) + \frac{1}{2}V(f,r) = \{\frac{1}{2}x + \frac{1}{2}y : |f(x)| < r, |f(y)| < r\} \subset V(f,r)$$

Therefore, if $f_1, ..., f_n \in X'$, and $r_1, ..., r_n > 0$, then $V = \{x : |f_j(x)| < r_j, 1 \leq j \leq n\} = \bigcap_{j=1}^n V(f_j, r_j)$. V is an element of the local base for τ'.

Therefore, $\frac{1}{2}V + \frac{1}{2}V = \bigcap_{j=1}^n \frac{1}{2}V(f_j, r_j) + \bigcap_{j=1}^n \frac{1}{2}V(f_j, r_j) \subset \bigcap_{j=1}^n V(f_j, r_j) = V$.

This shows that addition is coninuous.

Next, suppose $x \in X$, and α is a scalar. Then, $x \in sV$ for some $s > 0$, and V is as above. If $|\beta - \alpha| < r$, and $y - x \in rV$, then

$$\beta y - \alpha x = (\beta - \alpha)y + \alpha(y - x)$$

lies in V, provided that r is so small that $r(s + r) + |\alpha|r < 1$.

Hence, scalar multiplication is continuous

Lastly, we need to check that the space of all continuous linear functional is V'.

For one direction, by definition of initial topology, each $f \in X'$ is coninuous wrt τ'. Conversely, given a τ-CTS linear functional f on X. Then $f^{-1}B_1(0)$ is open, or there is U in the local base s.t $|f(U)| \subset [0, 1)$, and $ker(f) \subset U$. In particular, there are CTS linear functional $\{f_1, ..., f_n\} \subset X'$, and $r_1, ..., r_n > 0$ with

$$U = \{x \in X : |f_1(x)| < r_1, ...|f_n(x)| < r_n\}$$

and $\sup_{x \in U} |f(x)| \leq 1$.

Given $\epsilon > 0$, then

$$\begin{aligned} \epsilon U &= \{\epsilon x \in X : |f_j(x)| < r_j, 1 \leq j \leq n\} && (1) \\ &= \{\epsilon x \in X : |f_j(x)| < r_j, 1 \leq j \leq n\} && \text{(by linearity)} \\ &= \{y \in X : |f_j(y)| < \epsilon r_j, 1 \leq j \leq n\} && (\text{let } y = \epsilon x) \end{aligned}$$

This gives $\sup_{x \in \epsilon U} |f(x)| < \epsilon$. Taking intersection across all possible $\epsilon > 0$, we have:

$$\begin{aligned} ker(f) &= \bigcap_{\epsilon > 0} (\epsilon U) && (2) \\ &= ker(f_1) \cap ... \cap ker(f_n) && (3) \end{aligned}$$

Hence, by previous lemma, $f \in span\{f_1, ..., f_n\}$, so $f \in V'$.

This completes the proof. □

4.1.2 Definition. Let (X, τ) be a TVS whose dual X^* separates points. Then, the initial topology induced by X^* on X is called the weak topology τ_w.

4.1.3 Remarks.

- The weak topology τ_w is coarser than the topology τ we started with. Weak topology is useful when proving convergence in PDE, for example. Coarsening the topology will give us more room to find a candidate for limits. Furthermore, later on, we will see that weak* (weak-star) topology will give us compactness.
- When speaking of (X, τ_w), we abbreviate as X_w.
- In $\mathbb{R}^n$, the set of all linear functionals induces the standard topology in $\mathbb{R}^n$.

4.1.4 Remark. If τ' is a topology so that X^* is the set of τ'-CTS linear functional, then τ' is finer than τ_w. In other words, τ_w is the weakest topoloy on X that makes X into a locally convex space whose dual space is X^*.

4.1.5 Corollary. *X_w is locally convex, and ${X_w}^* = X^*$*

Proof. Since the dual X^* separates points, applying the preceding theorem, we get X_w is a locally convex space whose dual is X^*. Since τ_w is coarser than τ, all τ_w-CTS function is also τ-CTS . Thus, $X_w^* \subset X^*$.

On the other hand, τ_w is the coasest topology which X^* is cont, thus $X^* \subset X_w^*$. □

4.1.6 Corollary. $(X_w)_w = X_w$

Proof. Consider the space (X, τ_w). By the preceding theorem, $(\tau_w)_w$ is the topology that makes X into a locally convex space whose dual space is X^*, so $(\tau_w)_w \subset \tau_w$. However, τ_w is the coarsest topology with such property, so $\tau_w \subset (\tau_w)_w$. Hence, $(X_w)_w = X_w$. □

4.1.7 Proposition. *A sequence $(x_n)_{n\in\mathbb{N}}$ in a TVS weakly converges to zero, denoted by $x_n \overset{w}{\to} 0$, if and only if for each $f \in X^*, f(x_n) \to 0$.*

Proof. ($\Rightarrow$) Suppose $x_n \overset{w}{\to} 0$. Then, for each τ_w-neighborhood V of 0, there exists an $N \in \mathbb{N}$ such that for $n > N$, $x_n \in V$. Given $f \in X^*$, and $\epsilon > 0$, by weak convergence of $(x_n)_{n\in\mathbb{N}}$, there exists a constant $N \in \mathbb{N}$ s.t. for all $n \geq N, x_n \in V(f,\epsilon) = \{x : |f(x) < \epsilon\}$, so $|f(x_n)| < \epsilon$. Hence, $f(x_n) \to 0$.

($\Leftarrow$) Conversely, suppose $f(x_n) \to 0$. Given $\{f_1, f_2, ..., f_m\} \subset X^*$, and $r_1, r_2, ..., r_m > 0$, there exists constants $N_j \in \mathbb{N}, 1 \leq j \leq m$ so that $f_j(x_n) < r_j$ whenever $n > N_j$. Choose $N = max\{N_1, ..., N_m\}$, then, we have that $f_j(x_n) < r_j$, for all $j = 1, ..., m$, and $n > N$. In other words, $x_n \in V(f_1, r_1) \cap V(f_2, r_2) \cap ... \cap V(f_m, r_m)$, whenever $n \geq N$.

Then, given a neighborhood U of 0, we can find $\{f_1, f_2, ..., f_m\} \subset X^*$, and $r_1, r_2, ..., r_m$ such that the intersection is in U. Hence, $x_n \in U$, and $x_n \overset{w}{\to} 0$. □

4.1.8 Corollary. *Strong convergence (wrt τ) implies convergence wrt τ_w*

Proof. If $x_n \to 0$, then for $f \in X^*$, $f(x_n) \to 0$, so preceding proposition applies. □

4.1.9 Remark. The converse is not true. For example, consider space l^p of all functions x on positive integers such that

$$\Sigma_{n=1}^{\infty} |x(n)|^p < \infty$$

When $1 < p < \infty$, l^p contains sequences that converge weakly but not strongly.

4.2 Boundedness in the weak topology

4.2.10 Proposition. *Let (X, τ) be a TVS. A set E is τ_w-bounded if and only if for each linear functional $f \in X^*$, f is bounded on E*

Proof. E is weakly bounded iff for each neighborhood (in weak topology) V of 0, there exists an $s > 0$ s.t $t > s$, $E \subset tV$. Such neighborhood V is of the form $V = \{x : |f_j(x)| < r_j, 1 \leq j \leq n\}$, for $f_j \in X^*$, and $r_j > 0$.

Therfore, E is weakly bounded if and only if $E \subset \{tx \in X : |f_j(x)| < r_j, 1 \leq j \leq m\}$

Substituting $y = tx$, we obtain that $E \subset \{y \in X : |f_j(y)| < tr_j, 1 \leq j \leq m\}$.

Therefore, all f_j's are bounded on E. □

4.2.11 Proposition. *If (X, τ) is an infinite dimension TVS, then every τ_w-nbh of 0 contains an infinite dimensional subspace. In particular, (X, τ_w) is not locally bounded.*

Proof. Let U be an arbitrary neighborhood of 0, then there exists $V \subset U$ of the form

$$V = \{x \in X : |f_j(x)| < r_j, 1 \leq j \leq m\}$$

and let $N = \{x : f_1(x) = f_2(x) = ... = f_m(x) = 0\} = ker(f_1) \cap ker(f_2) \cap .... \cap ker(f_m)$.

We have that the map $x \mapsto (f_1(x), f_2(x), ..., f_m(x))$ has null space N, and $dim(X) \leq m + dim(N)$. Hence, $dim(N) = \infty$. Hence, X_w is not locally bounded □

4.3 Closedness in weak topology

4.3.12 Remark. If E is weakly closed, then by $E \subset \bar{E} \subset \bar{E}^w = E$, then E is closed in τ (this is because τ_w is coarser than the original topology τ, $\tau_w \subset \tau$, which gives us, $\bar{E} \subset \bar{E}^w$).

This brings up the question: When is $\bar{E}^w \subset \bar{E}$ true?