Functional Analysis, Math 7320
Lecture Notes from November 17, 2016

taken by Sabrine Assi

4.1.6 Theorem. *Suppose V is a vector space and V' is a separating vector space of linear functionals on V. Denote that τ' is the initial topology induced by V' on V. Then (V, τ') is locally convex topological space, and the space of all continuous linear functional is V'.*

Proof. Last time, we obtain local convex sub-base, and 0 is closed. Next for $f \in V'$ and if $r > 0$ we have $\frac{1}{2}V(f,r) + \frac{1}{2}V(f,r) = \left\{\frac{1}{2}x + \frac{1}{2}y : |f(x)| < r, |f(y)| < r\right\} \subseteq V(f,r)$.By the same argument with finite intersections, every element in local base formed from the sub-base satisfies the same inclusion under scaling and addition. this shows that addition is continuous at 0, and by translation invariant everywhere. By the definition of initial topology, each $f \in V'$ is continuous with respect to τ. Conversely, given a τ-continuous linear functional f on V.Then $f^{-1}(B_1(0))$ is open, or there is U in local base such that $|f(U)| \subseteq [0,1)$, and ker$f \subset U$. In particular, there are continuous linear functionals $\{f_1, ..., f_n\}, r_1....r_n > 0$ with

$$U = \{x \in V : |f_1(x)| < r_1, ..., |f_n(x) < r_n\}$$

and

$$sup_{x \in U} |f(x)| \leq 1$$

By linearity, $\epsilon U = \{x \in X : |f_1(x) < \epsilon r_1, ..., ..|f_n(x)| < \epsilon r_n\}$ gives

$$sup_{x \in \epsilon U} |f(x)| < \epsilon$$

Taking intersections,

$$ker f \subseteq \bigcap_{\epsilon > 0} (\epsilon U)$$

$$= ker(f_1) \cap ...... \cap ker(f_n)$$

Hence, by lemma, $f \in span\{f_1, ..., f_m\}$ so $f \in V'$ □

4.1.7 Example. In $\mathbb{R}^n$, let V' be the set of all linear functionals on $\mathbb{R}^n$. If $x \neq y$, there is $i \in \{1, ..., n\}$ such that $x_i \neq y_i$. Define $f_i(x_1, ..., x_i, ..., x_n) = x_i$. Then, $f_i \in V'$ and $x_i = f_i(x) \neq f(y) = y_i$. Thus, V' separates points in $\mathbb{R}^n$. Moreover, the initial topology induced by V' is the standard topology on $\mathbb{R}^n$.

The Weak Topology of a Topological Vector Space

Recall that the dual space X^* of X is the set of all continuous linear functionals on X, i.e.,

$$X^* = \{f : X \to \mathbb{K} : f \text{ is linear continuous}\}$$

where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

4.1.8 Remark. From the the previous theorem, if X^* separates points on X, the initial topology generated by X^* is a topological vector space. However, the dual space X^* might not separate points of X. For example, the space $L^p[0,1]$ where $0 < p < 1$ has only 0 as a continuous linear functional. The topology induced by continuous linear functional is not a topological vector space. We are interested in topological vector spaces. To make sure that the initial topology induced by X^* is a topological vector space, we need to include the condition of separating points of X^*.

Proof. To prove the example mentioned above for $0 < p < 1, L^p[0,1]^* = \{0\}$, we use contradiction. Assume there exists $\varphi \in L^p[0,1]^*$ with $\varphi \neq 0$.Then φ has image $\mathbb{R}$ (a nonzero linear map to a one - dimensional space is surjective),so there is some $f \in L^p[0,1]$ such that $|\varphi(f)| \geq 1$.Using this choice of f, map $[0,1]$ to $\mathbb{R}$ by

$$s \mapsto \int_0^s |f(x)|^p dx$$

This is continuous, so there is some s between 0 and 1 such that

$$\int_0^s |f(x)|^p dx = \frac{1}{2} \int_0^1 |f(x)|^p dx > 0.$$

Now let $g_1 = f_{\chi[0,s]}$ and $g_2 = f_{\chi(s,1]}$, so $f = g_1 + g_2$ and $|f|^p = |g_1|^p + |g_2|^p$. So

$$\int_0^1 |g_1(x)|^p dx = \int_0^s |f(x)|^p dx = \frac{1}{2} \int_0^1 |f(x)|^p dx,$$

hence $\int_0^1 |g_2(x)|^p dx = \frac{1}{2} \int_0^1 |f(x)|^p dx$. Since $|\varphi(f)| \geq 1$,$|\varphi(g_i)| \geq \frac{1}{2}$ for some i. Let $f_1 = 2g_i$, so $|\varphi(f_1)| \geq 1$ and $\int_0^1 |f_1(x)|^p dx = 2^p \int_0^1 |g_i(x)|^p dx = 2^{p-1} \int_0^1 |f(x)|^p dx$. Note that $2^{p-1} < 1$. by iteration, we get a sequence f_n in $L^p[0,1]$ such that $|\varphi(f_n)| \geq 1$ and

$$d(f_n, 0) = \int_0^1 |f_n(x)|^p dx = (2^{p-1})^n \int_0^1 |f(x)|^p dx \to 0,$$

a contradiction of continuity of φ.

□

4.1.9 Definition. Let (X, τ) be a topological vector space whose dual X^* separates points on X. Then the initial topology induced by X^* on X is called the weak topology and denoted by τ_w.

4.1.10 Corollary. *X_w is a locally convex topological vector space and $X_w^* = X^*$*

Proof. Since X^* separates points in X, X_w is a locally convex topological vector space by the previous theorem. Also, $X^* \subset X_w^*$ because weak star topology makes all functionals in X^* continuous w.r.t. τ_w. Conversely, original topology is finer than weak topology, thus all continuous function in weak topology is also continuous in the original topology. Therefore, $X_w^* \subset X^*$. $\square$

4.1.11 Corollary. $(X_w)_w = X_w$

4.1.12 Remark. As we know, the weak topology τ_w on X induced by X^* is the coarsest topology which any $f \in X^*$ is continuous. By the definition of X^*, any $f \in X^*$ is continuous with respect to the original topology τ. Therefore, $\tau_w \subseteq \tau$. Since a finite dimensional real (or complex) vector space is homeomorphism to $\mathbb{R}^n$ (or $\mathbb{C}^n$), the weak topology and the original topology in finite dimensional vector space are the same. The question that we should ask is that "in general, are the weak topology and the original topology in a topological vector space the same?", and "if they are not the same, do they share same properties?".

In general, if $f : X \to Y$ is continuous and $x_n \to x$, we have $f(x_n) \to f(x)$. Moreover, if X is first countable, we have that f is continuous if and only if $f(x_n) \to f(x)$ for every $x_n \to x$. Next theorem characterizes the weak convergence of a sequence in X by continuous linear functionls.

Warm up

4.1.13 Proposition. *A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological vector space weakly converges to zero, $x_n \overset{w}{\to} 0$, if and only if for each $f \in X^*$, $f(x_n) \to 0$*

Proof. For each τ_w -neighborhood of 0, there is $N \in \mathbb{N}$ such that for all $n \geq N$,$x_n \in V$. Given $f \in X^*$ and $\epsilon > 0$, then by weak convergence of $(x_n)_{n\in\mathbb{N}}$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in V(f, \epsilon)$, so $|f(x_n)| < \epsilon$ hence $f(x_n) \to 0$.Conversely, if $f(x_n) \to 0$ for each $f \in V^*$, then we have that for for each $\{f_1, ..., f_m\} \in X^*$ and $r_1, ...r_m$, there is $N \in \mathbb{N}$ for which if $n \geq N$, $x_n \in V(f_1, r_1) \cap V(f_2, r_2) \cap ...... \cap V(f_m, r_m)$ for every $U \in \mathcal{U}$,we can find $\{f_1, ..., f_m\}, r_1, ..., r_m$ such that the intersection is in U. Hence, we get $x_n \in U$ and $x_n \overset{w}{\to} 0$. $\square$

4.1.14 Corollary. *Strong convergence with respect to τ (i.e.convergence of a sequence with respect to τ) implies convergence with respect to τ_w.*

Proof. If $x_n \to 0$, then for $f \in X^*$, $f(x_n) \to 0$, so the preceding proposition applies. $\square$

4.1.15 Remark. The converse of the previous corollary is not true. For instance, in $L^p = L^p(-\pi, \pi)$ with respect to Lebesgue measure and $p \geq 1$. Define $f_n(t) = e^{int}$. Then, $f_n \to 0$ weakly in L^p but not strongly.

Next, we study boundedness in the weak topology

4.1.16 Proposition. *Let (X, τ) be a topological vector space. A set E is τ_w bounded if and only if for each linear functional $f \in X^*$,then f is bounded on E.*

Proof. E is weakly bounded if and only if for each τ_w-neighborhood V of 0, there exist $s \geq 0$ such that for all $t > 0$, $E \subset tV$.This is equivalent to the fact that for every set of the form $\{x \in X : |f_j(x)| < r_j, 1 \leq j \leq m\}$ and for all $t > s$,

$$E \subseteq \{tx \in X : |f(x)| < r_j, 1 \leq j \leq m\}$$

$$= \{y \in X : |f(y)| < tr_j, 1 \leq j \leq m\}$$

imply all $f_j's$ are bounded on E. $\square$

4.1.17 Proposition. *If (X, τ) is an infinite - dimensional topological vector space, then every τ_w -neighborhood of 0 contains an infinite dimensional subspace. In particular, (X, τ_w) is not locally bounded.(no chance for countable local base)*

Proof. Given $U \in \mathcal{U}$,then there exists $V \subset U$ of the form

$$V = \{x \in X : |f_j(x)| < r_j, 1 \leq j \leq m\}$$

and so is $N = kerf_1 \cap ...... \cap kerf_n$. Thus, $x \mapsto (f_1(x), f_2(x), ..., f_m(x))$ has the null space N.By dimension counting, $dimX \leq m + dimN$, this implies $dimN = \infty$. $\square$

Closedness in Weak Topology

4.1.18 Remark. If E is τ_w-closed, then by $E \subset \bar{E} \subset \bar{E}^w = E$. Then E is closed in τ.(this is because τ_w is coarser than the original topology).

Q: When is the reverse $E^w \subset E$ true ?