Functional Analysis, Math 7320
Lecture Notes from November 22, 2016

taken by Dylan Domel-White

2.1 Weak Topology vs. Original Topology (cont.)

Last time we defined the weak topology τ_w on a TVS X whose dual X^* separates points as the coarsest topology such that all elements of X^* are continuous. We then examined the relationships between the topological properties of (X, τ_w) and (X, τ). Regarding closedness, we remarked that because $\tau_w \subset \tau$, the τ-closure of any set E is contained in the τ_w-closure of E, i.e. $\overline{E} \subset \overline{E}^w$. Now we show that in a locally convex TVS, the reverse inclusion is also true.

2.1.1 Remark. The weak topology of a TVS X was only defined in the case that X^* separates points. However, recall that if X is a locally convex TVS, then X^* separates points (this was a corollary to Masur's theorem). Hence it always makes sense to talk about the weak topology on a locally convex TVS.

2.1.2 Theorem. *Let E be a convex subset of a locally convex TVS X. Then $\overline{E} = \overline{E}^w$.*

Proof. Recall a corollary to Masur's Theorem: if A and B are disjoint nonempty convex sets in a locally convex TVS with A compact and B closed, then there is a continuous linear functional f such that $\sup \text{Re } f(A) < \beta < \inf \text{Re } f(B)$ for some $\beta \in \mathbb{R}$.

Note that if $\overline{E} = X$, then since $\overline{E} \subset \overline{E}^w$ we have $\overline{E}^w = X$, hence $\overline{E} = \overline{E}^w$. So we proceed with the case when $\overline{E} \neq X$.

Let $x_0 \notin \overline{E}$. Since $\{x_0\}$ is compact and $\overline{E}$ is closed, there exists a continuous linear functional f such that Re $f(x_0) < \beta < \inf \text{Re } f(\overline{E})$ for some $\beta \in \mathbb{R}$. Then the set $\{x \in X : \text{Re } f(x) < \beta\}$ is a weakly open neighborhood of x_0 which does not intersect E. So $x_0 \notin \overline{E}^w$. Thus $\overline{E}^w \subset \overline{E}$ by taking complements. Since the reverse inclusion holds in general, we have equality. □

This theorem yields a simple corollary that helps us further understand closedness in the weak topology of a locally convex TVS.

2.1.3 Corollary. *For a convex subset E of a locally convex TVS:*

1. *E is τ-closed iff E is τ_w-closed.*
2. *E is τ-dense iff E is τ_w-dense.*

Proof. For (1), note that E is τ-closed iff $E = \overline{E}$. But this happens iff $E = \overline{E}^w$ since $\overline{E} = \overline{E}^w$ by the theorem. Finally, we have $E = \overline{E}^w$ iff E is τ_w-closed.

For (2), we have E is τ-dense iff $X = \overline{E}$. Again, using the theorem this happens iff $X = \overline{E}^w$ which is equivalent to saying E is τ_w-dense. □

In a metrizable space, we can characterize closed sets entirely in terms of sequences (since in this setting a point x is in $\overline{E}$ iff there exists a sequence of points in E which converges to x). Viewing the above theorem with this lens yields the following consequence for sequences.

2.1.4 Corollary. *If E is a convex set in a metrizable locally convex TVS X, and $(x_n)_{n\in\mathbb{N}}$ converges weakly to $x \in X$, then there is a sequence $(y_n)_{n\in\mathbb{N}} \subset E$ such that $y_n \to x$ in the original topology of X.*

Proof. Since $x_n \to x$ w.r.t. the weak topology, $x \in \overline{E}^w$. But by the above theorem, $\overline{E} = \overline{E}^w$, hence $x \in \overline{E}$. Thus since X is metrizable, there is a sequence $(y_n)_{n\in\mathbb{N}} \subset E$ which converges to x w.r.t. the original topology. □

2.2 The Weak-* Topology

2.2.5 Question. So far we have treated X^* as a vector space with no additional structure. If X is a Banach space, we could equip X^* with the operator norm to make it a Banach space as well. What should we do if X is a TVS with less structure?

2.2.6 Answer. We may use the linear functionals on X^*, i.e. the elements in X^{**} to define an initial topology on X^*. Recall from linear algebra that there is a natural identification of X^{**} with X given by the map $i : X \to X^{**}$ defined by $i(x) = F_x$, where $F_x(f) = f(x)$ for all $f \in X^*$ (F_x evaluates the functionals in X^* at x).

Note that $F_x(\alpha f + g) = (\alpha f + g)(x) = \alpha f(x) + g(x) = \alpha F_x(f) + F_x(g)$, so these maps are in fact linear functionals on X^*. We also know that $\{F_x\}_{x\in X}$ separates points in X^*, because if $f, g \in X^*$ and $f(x) = g(x)$ for all $x \in X$, then $f = g$. Hence $\{F_x\}_{x\in X}$ induces an initial topology on X^* called the *weak-* topology of X^**.

Note that for this definition we don't care if X^* separate points of X, since $\{F_x\}_{x\in X}$ always separates points of X^*.

2.2.7 Remark. In particular, the weak-* topology turns X^* into a locally convex TVS, and every weak-* continuous linear functional on X^* must actually equal F_x for some $x \in X$ (we proved this last time before defining the weak topology).

Since the weak-* topology on X^* is defined as an initial topology, we can characterize its open sets in terms of a local subbase of "balls".

2.2.8 Remark. The weak-* topology on X^* is generated by the local subbase $\{V(x, r)\}_{x\in X, r>0}$ where:

$$V(x, r) = \{f \in X^* : |F_x(f)| < r\} = \{f \in X^* : |f(x)| < r\}.$$

This characterization of weak-* open sets shows that weak-* convergence of a sequence of linear functionals is equivalent to pointwise convergence (in other words, the weak-* topology can be thought of as the *topology of pointwise convergence* on X^*).

2.2.9 Proposition. *Let X be a TVS, and X^* be equipped with the weak-* topology. Given a sequence $(f_n)_{n\in\mathbb{N}}$ in X^* and $f \in X^*$, then $f_n \to f$ iff for each $x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$.*

Proof. Suppose $f_n \to f$, i.e. $f_n - f \to 0$. Let $x \in X$ and fix $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that $f_n - f \in V(x, \epsilon)$ for all $n \geq N$. This means that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$. Since $\epsilon > 0$ was arbitrary, we have $f_n(x) \to f(x)$.

Conversely, suppose $f_n(x) \to f(x)$ for all $x \in X$. Let $U \subset X^*$ be a neighborhood of 0, then there exists a basis element $\bigcap_{i=1}^m V(x_i, r_i) \subset U$ which contains 0. For each $i = 1, \dots m$, there exists an $n_i \in \mathbb{N}$ such that $|f_n(x_i) - f(x_i)| < r_i$ for all $n \geq n_i$. Thus $f_n - f \in V(x_i, r_i)$ for all $n \geq n_i$. Letting $N = \max\{n_1, \dots, n_m\}$ we see that for all $n \geq N$, $f_n - f \in \bigcap_{i=1}^m V(x_i, r_i)$. Thus $f_n - f \to 0$, hence $f_n \to f$. □

We now examine an example to see how the weak-* topology relates to other ways of topologizing a dual space.

2.2.10 Example. Recall that $c_0^* = \ell_1$, and $\ell_1^* = \ell_\infty$. Since ℓ_1 is a dual space, we can equip it with the weak-* topology. However, we may also equip it with the operator norm inherited from c_0, or by the weak topology induced by its dual ℓ_∞.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in ℓ_1. We consider what it means for $x_n \to 0$ in these three different topologies:

- $(x_n)_{n\in\mathbb{N}} \longrightarrow 0$ w.r.t. $\|\cdot\|_1$ iff $\sum_{j=1}^\infty |(x_n)_j| \to 0$ as $n \to \infty$.
- $(x_n)_{n\in\mathbb{N}} \longrightarrow 0$ w.r.t. the weak topology iff for each $y \in \ell_\infty$:
$$\langle x_n, y\rangle = \sum_{j=1}^\infty (x_n)_j \overline{y_j} \to 0 \text{ as } n \to \infty.$$
- $(x_n)_{n\in\mathbb{N}} \longrightarrow 0$ w.r.t. the weak-* topology iff for each $y \in c_0$:
$$\langle x_n, y\rangle = \sum_{j=1}^\infty (x_n)_j \overline{y_j} \to 0 \text{ as } n \to \infty.$$

So we see that ℓ_1-convergence $\implies$ weak convergence $\implies$ weak-* convergence. We now consider whether the reverse implications are true for this example.

- Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in ℓ_1 defined by $(x_n)_j = -1$ if $j = n$, $(x_n)_j = -1$ if $j = n+1$, and $(x_n)_j = 0$ otherwise. Let $y \in c_0$, i.e. $y_j \to 0$. Note that $\langle x_n, y\rangle = (-1)^n(\overline{y_{n+1}} - \overline{y_n})$. Since $y_j \to 0$, we have $\overline{y_j} \to 0$, hence $\langle x_n, y\rangle \to 0$. Thus we have weak-* convergence of $(x_n)_{n\in\mathbb{N}}$ to 0.

  Now, let $z \in \ell_\infty$ be defined by $z_j = (-1)^j$. Then $\langle x_n, z\rangle = (-1)^n \cdot 2$ for all $n \in \mathbb{N}$. Hence $\langle x_n, z\rangle \not\to 0$. Thus (x_n) does not weakly converge to 0. This shows that weak-* convergence does not imply weak convergence for ℓ_1.

- It turns out that weak convergence in ℓ_1 *does* imply strong convergence, although this is not true for ℓ_p with $1 < p < \infty$. The proof is nontrivial, so see Conway's *A Course in Functional Analysis* for details (Rudin simply left it as an exercise).

Since the weak-* topology is so coarse, it has a nice compactness property: the closed unit ball of X^* is compact in the weak-* topology.

2.2.11 Theorem. (Banach-Alaoglu Theorem) *Let X be a topological vector space, $V \in \mathcal{U}$, and $K = \{f \in X^* : |f(x)| \leq 1 \; \forall x \in V\}$. Then K is weak-* compact.*

Proof. Note that V is absorbing since it is a neighborhood of 0, hence for each $x \in X$ there exists some $\beta(x) > 0$ such that $x \in \beta(x)V$, i.e. $\frac{1}{\beta(x)}x \in V$. Thus for $f \in K$, $f(\frac{x}{\beta(x)}) \leq 1$, hence $|f(x)| = \beta(x)|f(\frac{x}{\beta(x)})| \leq \beta(x)$.

For each $x \in X$, let $D_x = \{\alpha \in \mathbb{F} : |\alpha| \leq \beta(x)\}$, and define $P = \prod_{x \in X} D_x$ with the product topology. Note that each D_x is closed and bounded in $\mathbb{F}$, hence compact. So we have that P is compact by Tychonoff's theorem.

Note that every element in P is actually a function $f : X \to \mathbb{F}$ with the property that $|f(x)| \leq \beta(x)$ (these functions in P need not be linear). Since every $f \in K$ has this property, we see that $K \subset X^* \cap P$.

So K inherits two topologies: one from the weak-* topology on X^*, and one from the product topology on P. To proceed, we need to show that these topologies are actually the same.

2.2.12 Lemma. *The weak-* topology and product topology induced on K coincide.*

Proof. Let $f_0 \in K$. Choose any $x_i \in X$ for $1 \leq i \leq n$, and choose $\delta > 0$. Define the sets:

$$W_1 = \{f \in X^* : |f(x_i) - f_0(x_i)| < \delta \text{ for } 1 \leq i \leq n\}$$
$$W_2 = \{f \in P : |f(x_i) - f_0(x_i)| < \delta \text{ for } 1 \leq i \leq n\}.$$

Let n, x_i, and δ range over all possible values. Then W_1 forms a local base at f_0 for the weak-* topology of X^*, and W_2 forms a local base at f_0 for the product topology of P. Since $K \subset X^* \cap P$, we see that $W_1 \cap K = W_2 \cap K$, so both topologies that K inherits coincide. □

If we could now show that K is a closed subset of P, then that would mean K is compact with respect to the product topology. But since the product topology and weak-* topology on K are actually equivalent, we would have that K is weak-* compact.

2.2.13 Lemma. *K is a closed subset of P in the product topology.*

Proof. Let f_0 in the product-closure of K (we want to show $f_0 \in K$). For any $\alpha, \beta \in \mathbb{F}$, $x, y \in X$, and $\epsilon > 0$ we may define a neighborhood of 0:

$$S = \{f \in P : |f(x) - f_0(x)| < \epsilon, |f(y) - f_0(y)| < \epsilon, \text{ and } |f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \epsilon\}.$$

By f_0 being in $\overline{K}$, there is an $f \in K \cap S$, i.e. $|f(x) - f_0(x)| < \epsilon$, $|f(y) - f_0(y)| < \epsilon$, and $|f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \epsilon$. From these inequalities (and linearity of f) we get:

$$\begin{aligned}
&|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| \\
&\qquad = |f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y) + \alpha f(x) + \beta f(x) - \alpha f(x) - \beta f(x)| \\
&\qquad \leq |f_0(\alpha x + \beta y) - f(\alpha x + \beta y)| + |\alpha||f_0(x) - f(x)| + |\beta||f_0(y) - f(y)| \\
&\qquad \leq \epsilon + |\alpha|\epsilon + |\beta|\epsilon.
\end{aligned}$$

Since this holds for any $\epsilon > 0$ and any $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$, we have that f_0 is linear.

Lastly, for any $x \in V$ and $\epsilon > 0$, define $S' = \{f \in P : |f(x) - f_0(x)| < \epsilon\}$. Since S' is a neighborhood of f_0, there exists some $f \in K \cap S'$, i.e. $|f(x) - f_0(x)| < \epsilon$. Thus:

$$|f_0(x)| = |f_0(x) - f(x) + f(x)| \leq |f_0(x) - f(x)| + |f(x)|.$$

Since $f \in L$, we know $|f(x)| \leq 1$, hence $|f_0(x)| \leq \epsilon + 1$. Since $\epsilon > 0$ was arbitrary, we conclude that $|f_0(x)| \leq 1$. Thus $f_0 \in K$, so K is closed with respect to the product topology from P. □

This completes the proof. □