Functional Analysis, Math 7320
Lecture Notes from November 22, 2016

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4.2.5 Theorem. *If X is a locally-convex topological vector space and $E \subseteq X$ is convex, then the weak closure $\overline{E}^w$ of E is equal to its original closure $\overline{E}$.*

Proof. $\overline{E}^w$ is weakly closed, which implies that it is originally closed, which in turn implies that $\overline{E} \subseteq \overline{E}^w$. Conversely, let $x_0 \in X$ such that $x_0 \notin \overline{E}$. We use the following result of Hahn-Banach:

If A and B are disjoint, nonempty, convex subsets of a locally-convex topological vector space; A is compact; and B is closed, then there is a continuous linear functional f on X such that $\sup \operatorname{Re} f(A) < \inf \operatorname{Re} f(B)$.

This implies that there is a $\beta \in \mathbb{R}$ such that $\operatorname{Re} f(x_0) < \beta < \inf \operatorname{Re} f\left(\overline{E}\right)$. Therefore, $\{x \in X : \operatorname{Re} f(x) < \beta\}$ does not intersect E and is a weak open neighborhood of x_0, which implies that $x_0 \notin \overline{E}^w$, which in turn implies that $\overline{E}^w \subseteq \overline{E}$ after taking complements. □

4.2.6 Corollary. *If X is a metrizable, convex, topological vector space; $E \subseteq X$ is convex; and $(x_n)_{n\in\mathbb{N}}$ is a sequence in E that converges weakly to $x \in X$, then there is a sequence $(y_n)_{n\in\mathbb{N}}$ in E that converges originally to x.*

Proof. Let H be the convex hull of the set of all x_n and let K be the weak closure of H. Then $x \in K$. By the above theorem, x is in the original closure of H. Since the original topology is metrizable, there is a sequence $(y_n)_{n\in\mathbb{N}}$ in H that converges originally to x. □

4.2.7 Corollary. *For a convex subset E of a locally-convex topological vector space:*

(1) E is τ-closed if and only if E is τ_w-closed.

(2) E is τ-dense if and only if E is τ_w-dense.

Proof.

(1) The result follows from $\overline{E} = \overline{E}^w$.

(2) The result follows from (1): $\overline{E} = X$ if and only if $\overline{E}^w = X$.

□

4.3 The Weak-$*$ Topology on X^*

So far, we have considered X^* as a vector space. If X is normed, then we can equip X^* with the operator norm to turn it into a Banach space. However, what if X is a topological vector space? We will use a space of linear functionals on X to define a topology on X^*:

Consider $x \mapsto F_x$ on X, where $F_x(f) = f(x)$ for each $f \in X^*$. Then each F_x is linear because

$$F_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha F_x(f) + \beta F_x(g),$$

and $\{F_x\}_{x\in X}$ separates points in X^* because if $f(x) = g(x)$ for each $x \in X$, then $f = g$. Hence, $\{F_x\}_{x\in X}$ induces a topology on X^*. This topology is called the weak-$*$ topology.

4.3.8 Definition. The **weak-$*$ topology on** X^* is generated by the local subbase $\{v(x,r)\}_{x\in X, r>0}$, where $v(x,r) = \{f \in X^* : |f(x)| < r\}$. Hence, weak-$*$ convergence of a sequence $(f_n)_{n\in\mathbb{N}}$ to f means that for each $x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$.

4.3.9 Example. Recall that $c_0^* = \ell_1$ and $\ell_1^* = \ell^\infty$. Since ℓ_1 is a dual space, we can equip it with the weak-$*$ topology.

- If $(x_n)_{n\in\mathbb{N}} \xrightarrow{\|\cdot\|_1} 0$, then $\sum_{j=1}^\infty \left|(x_n)_j\right| \xrightarrow{n\to\infty} 0$.
- $(x_n)_{n\in\mathbb{N}} \xrightarrow{w} 0$ if and only if for each $y \in \ell^\infty$, $\langle x_n, y\rangle = \sum_{j=1}^\infty (x_n)_j \overline{y}_j \xrightarrow{n\to\infty} 0$.
- $(x_n)_{n\in\mathbb{N}} \xrightarrow{w^*} 0$ if and only if for each $y \in c_0$, $\langle x_n, y\rangle \xrightarrow{n\to\infty} 0$.

So ℓ_1 convergence implies weak convergence, which in turn implies weak-$*$ convergence.

Since the weak-$*$ topology is coarse, it has a nice compactness property:

4.3.10 Theorem. *Let (X,τ) be a topological vector space, let $V \in \mathcal{U}$, and let*

$$K = \{f \in X^* : |f(x)| \le 1 \text{ for all } x \in V\}.$$

Then K is weak-$*$ compact.

Proof. Since V is absorbing, for each $x \in X$, there is a $\beta(x) > 0$ such that $x \in \beta(x)V$. Thus for $x \in X$ and $f \in K$,

$$|f(x)| = \beta(x)\left|f\left(\frac{x}{\beta(x)}\right)\right| \le \beta(x).$$

Let $D_x = \{\alpha \in \mathbb{F} : |\alpha| \le \beta(x)\}$ and let $P = \prod_{x\in X} D_x$ be equipped with the product topology. Since each D_x is compact, Tychonoff's theorem implies that P is compact. Then every element of P is a function $f : X \to \mathbb{F}$ such that $|f(x)| \le \beta(x)$. This implies that every element of K is in P, which in turn implies that $K \subseteq X^* \cap P$.

We interject a lemma:

4.3.11 Lemma. *The weak-$*$ topology and the product topology induced on K coincide.*

Proof. Let $f_0 \in K$, let $x_1, \ldots, x_n \in X$, let $\delta > 0$, let

$$W_1 = \{f \in X^* : |f(x_j) - f_0(x_j)| < \delta \text{ for } 1 \le j \le n\}, \text{ and let}$$
$$W_2 = \{f \in P : |f(x_j) - f_0(x_j)| < \delta \text{ for } 1 \le j \le n\}.$$

Then as n, x_i, and δ range over all possible values, the resulting sets W_1 and W_2 form local bases for the weak-$*$ topology and the product topology at f_0 of X^* and P, respectively. However, since $K \subseteq X^* \cap P$, we have that $W_1 \cap K = W_2 \cap K$, which implies that both topologies restricted to K coincide. □

We interject another lemma:

4.3.12 Lemma. *K is a closed subset of P with respect to the product topology.*

Proof. Let f_0 be in the closure of K with respect to the product topology, let $x, y \in X$, let $\alpha, \beta \in \mathbb{F}$, and let $\varepsilon > 0$. Then

$$\begin{aligned} N = \{f \in P : &|f(x) - f_0(x)| < \varepsilon, \\ &|f(y) - f_0(y)| < \varepsilon, \text{ and} \\ &|f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \varepsilon\} \end{aligned}$$

is a neighborhood of f_0 in the product topology. Therefore, there is an $f \in K$ such that $f \in N$. Since f is linear, we have that

$$\begin{aligned} f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y) &= f_0(\alpha x + \beta y) - f(\alpha x + \beta y) \\ &\quad + f(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y) \\ &= (f_0 - f)(\alpha x + \beta y) + \alpha f(x) + \beta f(y) - \alpha f_0(x) - \beta f_0(y) \\ &= (f_0 - f)(\alpha x + \beta y) + \alpha(f - f_0)(x) + \beta(f - f_0)(y), \end{aligned}$$

which implies that

$$|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| < (1 + |\alpha| + |\beta|)\varepsilon.$$

Hence, f_0 is linear. If $x \in V$ and $\varepsilon > 0$, the same argument shows that there is an $f \in K$ such that $|f(x) - f_0(x)| < \varepsilon$. Since $|f(x)| \le 1$, by the definition of K, it follows that $|f_0(x)| \le 1$. As a result, $f_0 \in K$. □

This proves the theorem. □