Functional Analysis, Math 7320
Lecture Notes from December 1, 2016

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In these notes we study the relationship between compactness and total boundedness for convex hulls.

6.1.16 Theorem. *Suppose $A_1, \dots, A_n$ are compact, convex subsets of a topological vector space X. Then $\operatorname{co}(A_1) \cup \cdots \cup A_n)$ is compact. If X is locally convex and $E \subset X$ is totally bounded then $\operatorname{co}(E)$ is totally bounded. If X is a Frechét space and K is compact then $\overline{\operatorname{co}}(K)$ is compact. If $K \subset \mathbb{R}^n$ is compact then $\operatorname{co}(K)$ is compact.*

Proof. Let $S \subset \mathbb{R}^n$ be the simplex $\{(s_1, \dots, s_n) \colon s_i \geq 0, \sum s_i = 1\}$. Let $A = A_1 \times \cdots A_n$. Define $f \colon S \times A \to X$ by $f(s, a) = \sum s_i a_i$. Let $K = f(S, A)$. Then f is bilinear and hence continuous and $S \times A$ is compact, so K is compact. Next, if $a, b \in A$ with $a = (a_j)$, $b = (b_j)$, then for any $\lambda \in [0, 1]$ we have $\lambda a_j + (1 - \lambda) b_j \in A_j$ and hence $\lambda a + (1 - \lambda) b \in A$. Since f is linear in A, K is convex. Also $K \supset A_j$ for all j. Therefore $K \supset \operatorname{co}(A_1 \cup \cdots \cup A_n)$. Conversely, if $D \supset A_1 \cup \cdots \cup A_n$ is convex, then for all $s \in S$ and $a \in A$ we have $\sum s_i a_i \in D$ and so $K \subset D$. Hence $\operatorname{co}(A_1 \cup \cdots \cup A_n) \supset K$. Therefore $K = \operatorname{co}(A_1 \cup \cdots \cup A_n)$ is compact.

Choose $U \in \mathcal{O}(0)$. By local convexity there exists a convex open set $V \in \mathcal{O}(0)$ with $V + V \subset U$. Since E is totally bounded, let F be a finite subset of X such that $F + V \supset E$. Then $\operatorname{co}(F) + V \supset E$. Since $\operatorname{co}(F) + V$ is convex we have $\operatorname{co}(F) + V \supset \operatorname{co}(E)$. By the previous part, taking $F = \cup_{f \in F} \{f\}$ we know $\operatorname{co}(F)$ is compact. Next, $\operatorname{co}(F) + V = \cup_{x \in \operatorname{co}(F)} (x + F)$ is an open cover of F, so choose a finite set $F' \subset F$ such that $\operatorname{co}(F) \subset F' + V$. Then $\operatorname{co}(E) \subset \operatorname{co}(F) + V \subset F' + V + V \subset F' + U$ and so $\operatorname{co}(E)$ is totally bounded.

In a Frechét space, compactness is equivalent to being closed and totally bounded. Since K is compact, it is closed and totally bounded. By the previous part this means that $\operatorname{co}(K)$ is totally bounded. Therefore $\overline{\operatorname{co}}(K)$ is closed and totally bounded which means it is compact.

Let $S \subset \mathbb{R}^{n+1}$ be the convex simplex in $n+1$ dimensions. We are going to show that $\operatorname{co}(K) = f(S, A)$ as above with $A = K^{n+1}$. Let $x = \sum_{i=1}^{k+1} t_i x_i$ where $k > n$, $t_i \geq 0$, and $\sum t_i = 1$. Define $T \colon \mathbb{R}^{k+1} \to \mathbb{R}^n \times \mathbb{R}$ by $T(a) = (\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{k+1} a_i)$. Then $\dim \operatorname{Ran} T \geq 1$. And since $k > n$, T has a nontrivial kernel. Choose (a_j) such that $\sum_{j=1}^{n} a_j x_j = 0$ and $\sum_{j=1}^{k+1} a_j = 0$. Then there exists λ such that $|\lambda a_j| \leq t_j$ for at least one j. Writing $c_j = t_j - \lambda a_j$ gives $x = \sum_{j=1}^{k+1} c_j x_j$ where $c_j = 0$ for at least one j, while $\sum_{j=1}^{k+1} c_j = \sum_{j=1}^{k+1} t_j = 1$ and $c_j \geq 0$. This means we have removed a term from the sum defining x. Inductively remove terms while $k > n$ and so we can reduce the number of terms to $n + 1$ as claimed. $\square$

6.1.17 Definition. Suppose K is a subset of a vector space V. A point $z \in K$ is called an *extreme point* of K if for any $x, y \in K$, if $z = \lambda x + (1-\lambda)y$ where $\lambda \in (0,1)$ then $x = y = z$. Denote the set of all extreme points of K by $E(K)$.

6.1.18 Definition. A nonempty set $S \subset K$ is an extreme set of K if for any $x, y \in K$, if $\lambda x + (1-\lambda)y \in S$ and $\lambda \in (0,1)$ implies that $x, y \in S$.

6.1.19 Theorem (Krein-Milman Theorem)**.** *If X is a topological vector space for which X^* separates points and $K \subset X$ is nonempty, compact, and convex, then $K = \overline{\text{co}}(E(K))$.*

Proof. Let P be the set of all compact extreme sets in K. Then $K \in P$ so $P \not\emptyset$. Next, if $P' \subset P$ and S is the intersection of all sets in P', then $S \subset P$, for if $\lambda x + (1-\lambda)y \in A$ and $x, y \in K$ with $\lambda \in (0,1)$, then $x, y \in A$ for all $A \in P'$. Thus $x, y \in S$ and S is compact. Now choose $S \in P$ and $f \in X^*$ and define $S_f = \{x \in S \colon \Re f(x) = \max_{x \in S} \Re f(x)\}$. In other words S_f is the set of points in S at which f attains its maximum value. Since S is compact, S_f is nonempty. And S compact implies S_f compact since S_f is a closed subset of S by the continuity of f. Also, S_f is convex and it can be shown that S_f is an extreme set (this follows easily from the fact that f is linear)which means that $S_f \in P$.

To be continued... □