Functional Analysis, Math 7320
Lecture Notes from December 1, 2016

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We examine the interplay between compactness and total boundedness for the convex hull.

2.9.9 Proposition. *If $K \subset \mathbb{R}^n$ and $x \in co(K)$, then x lies in the convex hull of some subset of K which contains at most $n+1$ points.*

Proof. It is enough to show that if $k > n$ and $x = \sum_{j=1}^{k+1} t_j x_j$ is a convex combination of some $k+1$ vectors $x_j \in \mathbb{R}^n$, then x is actually a convex combination of some k of these vectors. Assume, with no loss of generality, that $t_j > 0$ for $1 \leq j \leq k+1$. The null space of the linear map

$$(a_1, \dots, a_{k+1}) \mapsto \left(\sum_{j=1}^{k+1} a_j x_j, \sum_{j=1}^{k+1} a_j \right),$$

which sends $\mathbb{R}^{k+1}$ into $\mathbb{R}^n \times \mathbb{R}$, has positive dimension, since $k > n$. Hence, there exists $(a_1, \dots, a_{k+1})$, with some $a_j \neq 0$, so that $\sum_{j=1}^{k+1} a_j x_j = 0$ and $\sum_{j=1}^{k+1} a_j = 0$. Since $t_j > 0$ for all j, there is a constant λ such that $|\lambda a_j| \leq t_j$ for all j and $\lambda a_j = t_j$ for at least one j. Setting $c_j = t_j - \lambda a_j$, we conclude that $x = \sum_{j=1}^{k+1} c_j x_j$ and that at least one c_j is 0; note also that $\sum_{j=1}^{k+1} c_j = \sum_{j=1}^{k+1} t_j$ and that $c_j \geq 0$ for all j. ☐

2.9.10 Theorem. *Let $A_1, A_2, \dots, A_n$ be compact convex subsets of a topological vector space X. Then,*

(a) the convex hull $co(A_1 \cup \ldots \cup A_n)$ is compact.

(b) If X is locally convex and if $E \subset X$ is totally bounded, then $co(E)$ is totally bounded.

(c) If X is a Frechét space (so X is locally convex, metrizable, complete) and if $K \subset X$ is compact, then $\overline{co}(K)$ is compact.

(d) If $K \subset \mathbb{R}^n$ is compact, then $co(K)$ is compact.

Proof. (a) Let $S \subset \mathbb{R}^n$ be the simplex

$$S = \left\{ (s_1, s_2, \dots, s_n) : s_j \geq 0, \sum_{j=1}^{n} s_j = 1 \right\}$$

and for $A = A_1 \times A_2 \times \ldots \times A_n$, let $f : S \times A \to X$ be given by

$$f(s,a) = \sum_{j=1}^{n} s_j a_j.$$

Consider $K = f(S,A)$. Then, by continuity of f and compactness of S and A, K is compact. Next, if $a = (a_j)_{j=1}^n$ and $b = (b_j)_{j=1}^n$, then by convexity of each A_j, for each $a_j, b_j \in A_j$ and $\lambda \in [0,1]$, we get $\lambda a_j + (1-\lambda)b_j \in A_j$. Thus, $a, b \in A$ gives $\lambda a + (1-\lambda)b \in A$, so by linearity of f in A, K is convex. Moreover, $A_j \subset K$ for each $j = \{1, 2, \ldots, n\}$ and thus $co(A_1 \cup A_2 \cup \ldots \cup A_n) \subset K$. Finally, if D is convex and $A_1 \cup A_2 \cup \ldots \cup A_n \subset D$, then for each $s \in S$, $a \in A$, we get $\sum_{j=1}^n s_j a_j \in D$, so $K \subset D$. Thus,

$$co(A_1 \cup A_2 \cup \ldots \cup A_n) = \bigcap_{D \text{ convex},\ A_j \subset D} D = K.$$

(b) Let $U \in \mathcal{U}$. By local convexity, there is a convex open (balanced) $V \in \mathcal{U}$, with $V + V \subset U$. From the total boundedness of E, let $F \subset X$, with $|F| < \infty$, such that $E \subset F + V$. So $E \subset co(F) + V$ and by taking the convex hull on the left hand side, we obtain

$$co(E) \subset co(F) + V.$$

From (a), with $F = \cup_{j=1}^n A_j$, where $A_j = \{x_j\}$, $x_j \in X$, we see $co(F)$ is compact, and from

$$co(F) + V = \bigcup_{x \in co(F)} (x + V)$$

being an open cover of F, there exists F', with $|F'| < \infty$, such that $F' \subset co(F)$ with $co(F) \subset F' + V$ and thus

$$co(E) \subset co(F) + V \subset F' + V + V \subset F' + U,$$

which means $co(E)$ is totally bounded.

(c) In a complete metric space, closed and totally bounded sets are compact and vise versa. Hence, by the assumption that K is compact, it is also totally bounded, and by (b), the same holds for $co(K)$. Finally, $\overline{co}(K)$ is totally bounded (and closed), and so $\overline{co}(K)$ is compact.

(d) Let S be the simplex in $\mathbb{R}^{n+1}$ consisting of all $t = (t_1, \ldots, t_{n+1})$ with $t_j \geq 0$ and $\sum_{j=1}^{n+1} t_j = 1$. Let K be compact, $K \subset \mathbb{R}^n$. By the previous proposition, $x \in co(K)$ if and only if

$$x = \sum_{j=1}^{n+1} t_j x_j$$

for some $t \in S$ and $x_j \in K$. In other words, $co(K)$ is the image of $S \times K^{n+1}$ under the continuous mapping

$$(t, x_1, \ldots, x_{n+1}) \mapsto \sum_{j=1}^{n+1} t_j x_j.$$

Hence $co(K)$ is compact. $\square$

2.10 Extreme points

2.10.11 Definition. Let V be a vector space and let $K \subset V$. A point $z \in K$ is called an extreme point of K if for $x, y \in K$, with $z = \lambda x + (1-\lambda)y$, $0 < \lambda < 1$, we have $x = y = z$. We denote the set of extreme points as $E(K)$. More generally, a non-empty set $S \subset K$ is called an extreme set if for $x, y \in K$, $0 < \lambda < 1$, with $\lambda x + (1-\lambda)y \in S$ we have $x, y \in S$.

2.10.12 Theorem. *(Krein-Milman) Let X be a topological vector space on which X^* separates points. If $K \subset X$ is a non-empty compact convex set in X, then $K = \overline{co}(E(K))$.*

Proof. Let P be the collection of all compact extreme sets of K. Since $K \in P$, $P \neq \emptyset$. We shall use the following properties of P:

1. The intersection S of any non-empty subcollection of P is a member of P, unless $S = \emptyset$.

2. If $S \in P$, $\Lambda \in X^*$, μ is the maximum of $Re(\Lambda)$ on S, and

$$S_\Lambda = \{x \in S : Re(\Lambda x) = \mu\},$$

   then $S_\Lambda \in P$.

The proof of (1) is immediate. To prove (2), suppose $\lambda x + (1-\lambda)y = z \in S_\Lambda$, $x \in K$, $y \in K$, $0 < \lambda < 1$. Since $z \in S$ and $S \in P$, we have $x \in S$ and $y \in S$. Hence $Re(\Lambda x) \leq \mu$. Since $Re(\Lambda z) = \mu$ and Λ is linear, we conclude: $Re(\Lambda x) = \mu = Re(\Lambda y)$. Hence $x \in S_\Lambda$ and $y \in S_\Lambda$. This proves (2).

Choose some $S \in P$. Let P' be the collection of all members of P that are subsets of S. Since $S \in P'$, P' is not empty. Partially order P' by set inclusion, let Ω be a maximal totally ordered subcollection of P', and let M be the intersection of all members of Ω. Since Ω is a collection of compact sets with the finite intersection property, $M \neq \emptyset$. By (1), $M \in P'$. The maximality of Ω implies that no proper subset of M belongs to P. It now follows from (2) that every $\Lambda \in X^*$ is constant on M. Since X^* separates points on X, M has only one point. Therefore M is an extreme point on K. We have now proved that $E(K) \cap S \neq \emptyset$ for every $S \in P$. In other words, every compact extreme set of K contains one extreme point of K.

Since K is compact and convex, we have $\overline{co}(E(K)) \subset K$ and this shows that $\overline{co}(E(K))$ is compact. Assume, to reach a contradiction, that some $x_0 \in K$ is not in $\overline{co}(E(K))$. The previous theorem gives us a $\Lambda \in X^*$ such that $Re(\Lambda x) < Re(\Lambda x_0)$ for every $x \in \overline{co}(E(K))$. If K_Λ is defined as in (2), then $K_\Lambda \in P$. Our choice of Λ shows that K_Λ is disjoint from $\overline{co}(E(K))$, and this contradicts $E(K) \cap S \neq \emptyset$. □

2.10.13 Example. $L^1(0,1)$ is not the dual of any space X.

Proof. Suppose towards a contradiction that $L^1 = X^*$. Then consider $B_1^{L^1}$, which is compact in the weak-$*$ topology. However, we note that $E(B_1^{L^1}) = \emptyset$. Also note that given any $f \in B_1^{L^1}$, we can write

$$f = f\chi_{[0,a]} + f\chi_{[a,1]},$$

where a satisfies

$$\int_0^a |f| = \int_a^1 |f| = \frac{\|f\|_{L^1}}{2}.$$

Such an a can be found, since $\int_0^x |f|$ is continuous in x. Then we have that

$$f = \frac{1}{2}(2f\chi_{[0,a]}) + \frac{1}{2}(2f\chi_{[a,1]}),$$

where $\|2f\chi_{[0,a]}\|_{L^1} = \|2f\chi_{[a,1]}\|_{L^1} = \|f\|_{L^1}$, i.e. both parts are in $B_1^{L^1}$ and so f is not an extreme point. Now by Krein-Milman, we have that $B_1^{L^1}$ is the closure of the convex hull of its extreme points, but since the set of extreme points is empty, this is a contradiction. Thus, L^1 cannot be the dual of any space. $\square$