Functional Analysis, Math 7320
Lecture Notes from December 01, 2016

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We explore the interplay between compactness and total boundedness for the convex hull.

0.0.1 Theorem. *Let $A_1, A_2, ..., A_n$ be compact convex subsets of a TVS X. Then*

1. *$co(A_1 \cup A_2 \cup ... \cup A_n)$ is compact*
2. *if X is locally convex and $E \subset X$ is totally bounded, then $co(E)$ is totally bounded*
3. *if X is a Fréchet space (i.e. locally convex, metrizable and complete) and $K \subset X$ is compact then $\overline{co}(K)$ is compact*
4. *if $K \subset \mathbb{R}^n$ is compact, then $co(K)$ is compact*

Proof. 1. Let $S \subset \mathbb{R}^n$ be the simplex, i.e.

$$S = \{(s_1, s_2, ..., s_n) : s_j \geq 0, \sum_{j=1}^{n} s_j = 1\}$$

and for $A = A_1 \times A_2 \times ... \times A_n$, let $f : S \times A \longrightarrow X$ defined by

$$f(s, a) = \sum_{j=1}^{n} s_j a_j$$

Consider $K = f(S, A)$. By continuity of f and compactness of S and A, K is compact. Next, if $a = (a_j)_{j=1}^n$ and $b = (b_j)_{j=1}^n$ then by convexity of each A_j, for any $a_j, b_j \in A_j$ and $\lambda \in [0, 1]$

$$\lambda a_j + (1 - \lambda) b_j \in A_j$$

So, $a, b \in A$ gives that $\lambda a + (1 - \lambda) b \in A$ hence, by linearity of f in A, K is convex. Moreover, $K \supset A_j$ for each $j = 1, 2, ..., n$. But $co(A_1 \cup A_2 \cup ... \cup A_n)$ is the smallest convex set containing all A_j, thus, $K \supset co(A_1 \cup A_2 \cup ... \cup A_n)$.
Finally, if D is convex and $D \supset (A_1 \cup A_2 \cup ... \cup A_n)$ then for each $s \in S$ and $a \in A$ $\sum_{j=1}^{n} s_j a_j \in D$ so $D \supset K$. Thus,

$$\begin{aligned} co(A_1 \cup A_2 \cup ... \cup A_n) &= \bigcap_{\substack{D \text{ convex} \\ D \supset A_j}} D \\ &= K \end{aligned}$$

2. Let $U \in \mathcal{U}$. By local convexity, there exists some convex, open (balanced) $V \in \mathcal{U}$ such that $V + V \subset U$. From total boundedness of E, let $F \subset X$, $|F| < \infty$, $F + V \supset E$. So $co(F) + V \supset E$. Then, since the LHS is a convex set containing E and $co(E)$ is the smallest convex containing E, we get

$$co(F) + V \supset co(E)$$

From (1), assuming $F = \bigcup\limits_{j=1}^{n} A_j$, $A_j = \{x_j\}$ and $x_j \in X$, $co(F)$ is compact, and from

$$co(F) + V = \bigcup_{x \in co(F)} (x + V)$$

being an open cover of F there exists some F' such that $|F'| < \infty$, $F' \subset co(F)$ and with $co(F) \subset F' + V$. Thus

$$\begin{aligned} co(E) &\subset co(F) + V \\ &\subset F' + V + V \\ &\subset F' + U \end{aligned}$$

Hence, $co(E)$ is totally bounded.

3. In a complete metric space, closed and totally bounded sets are compact and vice versa. Hence, by the assumption that K is compact, it is also totally bounded and by (2) so is $co(K)$. But then, the closure $\overline{co}(K)$ is totally bounded (and closed) so, by our initial remark, $\overline{co}(K)$ is compact.

4. As above, let $S \subset \mathbb{R}^{n+1}$ be the convex simplex in $n+1$ dimensional Euclidean space. We will show that for a compact subset $K \subset \mathbb{R}^n$

$$co(K) = f(S, A)$$

where $A = K^{n+1}$, the direct product of $n+1$ copies of K.
Let

$$x = \sum_{j=1}^{k+1} t_j x_j$$

where $k > n$, $t_j \geq 0$ and $\sum\limits_{j=1}^{k+1} t_j = 1$. We will show that there exists some proper subset of $\{x_1, x_2, ..., x_{k+1}\}$ such that x is a convex combination of elements in that subset.
Assume that $t_j > 0$, $x_i \neq 0$ for all j (otherwise we could get the proper subset simply by discarding all such x_j's from the set). Consider the mapping $T : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}^n \times \mathbb{R}$ defined as

$$T(a) = \left(\sum_{j=1}^{k+1} a_j x_j, \sum_{j=1}^{k+1} a_j \right)$$

Then, being a linear map, we have $dimrange(T) \geq 1$. From rank-nullity and $k > n$, there exists some $a \in \mathbb{R}^{k+1}$ such that $\sum_{j=1}^{k+1} a_j x_j = 0$ and $\sum_{j=1}^{k+1} a_j = 0$. By our assumption that $t_j > 0$, there exists some $\lambda \in \mathbb{R}$ such that $|\lambda a_j| \leq t_j$ for all $j = 1, 2, ..., k+1$ and $\lambda a_j = t_j$ for at least one value of j. Setting

$$c_j = t_j - \lambda a_j$$

gives that

$$x = \sum_{j=1}^{k+1} c_j x_j$$

and $c_j = 0$ for at least one value of $j = 1, 2, ..., k+1$, while

$$\sum_{j=1}^{k+1} c_j = \sum_{j=1}^{k+1} T_j = 1, \ \ c_j \geq 0$$

Proceeding inductively, we can remove terms as long as $k > n$, reducing the number of terms to $n + 1$ as claimed.

□

Extreme Points

0.0.2 Definition. 1. Let V be a vector space and $K \subset V$. A point $z \in K$ is called an *extreme point* of K if for any pair $x, y \in K$ such that $z = \lambda x + (1 - \lambda) y$, $\lambda \in [0, 1]$, we have $x = y = z$. We will denote the set of all extreme points of K as $E(K)$.

2. More generally, a non-empty set $S \subset K$ is called an *extreme set* if for any pair $x, y \in K$ we have

$$\lambda x + (1 - \lambda) y \in S, \lambda \in [0, 1] \Longrightarrow x, y \in S$$

0.0.3 Theorem (*Krein-Milman*). *Let X be a TVS for which X^* separates points. If $K \subset X$ is non-empty compact and convex, then*

$$K = \overline{co}(E(K))$$

Proof. Let $\mathcal{P}$ be the collection of all compact extreme subsets of K. Then $\mathcal{P} \neq \emptyset$ because $K \in \mathcal{P}$.
Claim 1: The intersection S of any non-empty sub-collection of sets in $\mathcal{P}$ is also a member of $\mathcal{P}$, unless $S = \emptyset$.
Indeed, if $\mathcal{P}' \subset \mathcal{P}$ and $\emptyset \neq S = \bigcap_{A \in \mathcal{P}'} A$, then $S \in \mathcal{P}$ because if $x, y \in K$ are such that $\lambda x + (1 - \lambda) y \in S$ for $\lambda \in [0, 1]$ then $x, y \in A$ for all $A \in \mathcal{P}'$, thus $x, y \in S$. Moreover, S is compact.
Claim 2: If $S \in \mathcal{P}$ and $f \in X^*$ and

$$S_f = \{x \in S : Ref(x) = \mu\}$$

with $\mu = \max_{x \in S} Ref(x)$, then $S_f \in \mathcal{P}$.
Indeed, assume again that $x, y \in K$ such that $\lambda x + (1-\lambda)y = z \in S_f$ for $\lambda \in [0,1]$. Hence $Ref(x) \leq \mu$ and $Ref(y) \leq \mu$ by definition of μ. If it was the case that $Ref(x) < \mu$ or $Ref(y) < \mu$, by linearity of f we would get

$$\begin{aligned} \lambda Ref(x) + (1-\lambda)Ref(y) &< \lambda\mu + (1-\lambda)\mu \\ Re(\lambda f(x) + (1-\lambda)f(y)) &< \mu \\ Ref(z) &< \mu \end{aligned}$$

which contradicts our assumption that $Ref(z) = \mu$. We conclude that $Ref(x) = \mu = Ref(y)$, i.e. $x, y \in S_f$.
Now, choose some $S \in \mathcal{P}$ and let $\mathcal{P}'$ be the collection of all members of $\mathcal{P}$ that are subsets of S. Observe that $\mathcal{P}' \neq \emptyset$ because $S \in \mathcal{P}'$. Partially order $\mathcal{P}'$ by set inclusion, let Ω be a maximal totally ordered sub-collection of $\mathcal{P}'$ and let M be the intersection of all members of Ω. Since Ω is a collection of compact sets with the finite intersection property, $M \neq \emptyset$ and since it is the intersection of a sub-collection of sets in $\mathcal{P}'$, $M \in \mathcal{P}'$ by Claim 1. The maximality of Ω implies that no proper subset of M belongs to $\mathcal{P}$. From Claim 2 we get that every $f \in X^*$ is constant on M. Since X^* separates points on X, M has only one point. Therefore M is an extreme point of K.
We have now proved that

$$E(K) \cap S \neq \emptyset$$

for every $S \in \mathcal{P}$. i.e. every compact extreme set of K contains one extreme point of K.
Since K is compact and convex, we have

$$\overline{co}(E(K)) \subset K$$

which shows that $\overline{co}(E(K))$ is compact.
For the inverse relation we will need the following:
Claim 3: If A and B are disjoint, nonempty, compact, convex sets in X then there exists some $f \in X^*$ such that

$$\sup_{x \in A} Ref(x) < \inf_{y \in B} Ref(y)$$

(the proof can be found in Rudin's book - Theorem 3.21 pg 74)
Now, if it was the case that some $x_0 \in K$ was not in $\overline{co}(E(K))$, Claim 3 then implies the existence of an $f \in X^*$ such that $Ref(x) < Ref(x_0)$ for every $x \in \overline{co}(E(K))$. Letting K_f denote again the set

$$K_f = \{x \in K : Ref(x) = \mu\}$$

where $\mu = \max_{x \in K} Ref(x)$, then $K_f \in \mathcal{P}$. This choice of f shows that K_f is disjoint from $\overline{co}(E(K))$, hence contradicts $E(K) \cap S \neq \emptyset$. We conclude that $K \subset \overline{co}(E(K))$, which completes the proof.
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