Fundamental Theorems from Functional Analysis
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1 Consequences of Completeness

The following theorems are consequences of Baire's theorem. In the context of Banach spaces, the most fundamental property implied by Baire's theorem is that the interior of the closed unit ball is non-empty. This property is essential for the proofs of the theorems listed hereafter.

1.1 Uniform boundedness

The first main insight is concerned with uniform boundedness, a special case of equicontinuity when a family of bounded linear maps is concerned.

1.1 Theorem. *(Uniform boundedness) Let Γ be a collection of continuous linear maps from a Banach space X to a normed vector space Y. If for each $x \in X$,*

$$\sup_{A \in \Gamma} \|Ax\| < \infty,$$

then Γ is uniformly bounded, meaning $\sup_{A\in\Gamma} \|A\| < \infty$.

1.2 Operators defined as limits of sequences

Next, we state a consequence for sequences of operators.

1.2 Corollary. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of continuous linear maps from a Banach space X to a normed vector space Y, and suppose for all $x \in X$, $A(x) = \lim_{n\to\infty} A_n x$ exists, then $\sup_n \|A_n\| < \infty$, and A is a bounded linear map.*

If the uniform boundedness is known and assuming Y is complete, then the sequence only needs to be defined on a dense set in X.

1.3 Corollary. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of continuous linear maps from a Banach space X to a Banach space Y with $\sup_n \|A_n\| < \infty$, and suppose for all $x \in D$, where D is a dense set, $A(x) = \lim_{n\to\infty} A_n x$ exists, then $A_n x$ converges for each $x \in H$ and defines a bounded linear map.*

1.3 Operators having bounded inverses

Let X, Y be two normed vector spaces. We say $f : X \to Y$ is open at point $p \in X$ if $f(V)$ contains a neighborhood of $f(p)$ whenever V is a neighborhood of p. We say that f is open if $f(U)$ is open in Y whenever U is open in X. A linear mapping between two topological vector spaces is open if and only if it is open at the origin.

1.4 Theorem. *Let X and Y be Banach spaces and A a bounded linear map from X to Y. If A is onto, then A is an open mapping.*

This has a consequence for inverse maps.

1.5 Theorem. *Let X and Y be Banach spaces and A a bounded linear map from X to Y. If A is onto and one-to-one, then the inverse of A is a bounded linear map.*

More concretely we can formulate this in terms of the norms on X and Y.

1.6 Corollary. *If X and Y are Banach spaces and A a bounded linear map from X to Y. If A is onto and one-to-one, then there exist $m, M > 0$ with*

$$m\|x\|_X \leq \|Ax\|_Y \leq M\|x\|_X.$$

2 Convexity and Linear Functionals

When computing with vectors, we need to introduce coordinates. In finite dimensions, we typically choose a basis for the space of linear functionals. For infinite dimensional vector spaces, we prove that the set of linear functionals distinguishes between any two distinct vectors. The main tool to establish this is the Hahn-Banach theorem. There is a version for real vector spaces.

2.1 Theorem. *Let V be a real vector space and $p : V \to \mathbb{R}$ a function satisfying*

- *$p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$, and*
- *$p(\alpha x) = \alpha p(x)$ for all $x \in V$ and $\alpha \in \mathbb{R}^+$.*

Let $X \subseteq V$ be a (linear) subspace and f a linear functional $f : X \to \mathbb{R}$ such that $f \leq p|_Y$, then there is a linear functional $F : V \to \mathbb{R}$ such that $F|_Y = f$ and $F \leq p$ on all of V

When the conditions on p are restricted, we can establish a version or complex vector spaces.

2.2 Theorem. *Let V be a complex vector space and p a seminorm on V. If $X \subseteq V$ is a subspace, $f : X \to \mathbb{C}$ a linear functional satisfying $|f| \leq p|_X$, then there is a linear functional $F : V \to \mathbb{C}$ with $F|_X = f$ and $|F| \leq p$ on all of V.*