Lecture Notes from August 23, 2022

taken by Bernhard Bodmann

0 Course Information

Class:	Tu&Th 10am-11:20pm, Melcher Hall 129
Instructor:	Bernhard Bodmann, *bgb*@math.uh.edu
Office:	PGH 604; Tu 1:30-3pm, We 10:30-11:30am
Content:	This course is part of a two semester sequence covering main advanced results in functional analysis, including Hilbert spaces, Banach spaces, and linear operators on theses spaces. Functional analysis combines two fundamental branches of mathematics: analysis and linear algebra. Limiting arguments from analysis become essential in order to resolve questions from linear algebra in infinite-dimensional spaces. In addition, there are close connections between algebraic and topological properties in such spaces that deepen our understanding even in the finite dimensional case. Topics covered in this first part of the course sequence include: A review of fundamental concepts, including Hilbert spaces and operators on Hilbert spaces, the adjoint operator, normal operators. Semigroups with involution and their representation, decomposition of representations, spectral theory in Banach algebras, C^* algebras, properties of the spectrum, Gelfand's representation theory, properties of commutative C^*-algebras, functional calculus, positivity, states, spectral theory for bounded normal operators.
Prerequisites:	Apart from the official prerequisites, you should have seen the Hahn Banach Theorem, the Uniform Boudnedness Theorem together with its consequence for operator sequences, the Banach-Steinhaus Theorem, and the Open Mapping Theorem, all in the context of normed spaces. For details, see the course handout from August 23, 2022.
Text:	Walter Rudin, Functional Analysis, 2nd edition, McGraw Hill, 1991.
Assignments:	You will be asked take notes and typeset them in LaTeX.

Final Grade: Based on the quality of notes. The goal of the notes is to absorb the material presented in class and prepare the notes so that one of your (hypothetical) peers who missed class will be able to follow your explanations and learn what happened in class.

All of the course-related information is listed in the official syllabus, which can be found on the website for our course:

www.math.uh.edu/~bgb/Courses

1 Fundamental concepts

We begin with a review of fundamental concepts. Most of the material in this course is based on complex Hilbert spaces. In many cases, a real version can be extracted from the complex result without much effort.

1.1 Definition. Let $\mathcal{H}$ be a complex vector space.

(a) A map $b : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is called

(1) *sesquilinear*, if fixing the second entry $y \in \mathcal{H}$ gives a linear map $x \mapsto b(x, y)$, and fixing the first entry $x \in \mathcal{H}$ gives a map $y \mapsto b(x, y)$ that is *conjugate linear*, so for $\lambda \in \mathbb{C}$, $x, y, z \in \mathcal{H}$,

$$b(x, \lambda y + z) = \overline{\lambda} b(x, y) + b(x, z)\,.$$

Unless indicated otherwise, $\overline{\lambda}$ always denotes the complex conjugate of a complex number λ.

(2) The map b is called *Hermitian* if b is sesquilinear and additionally $b(y, x) = \overline{b(x, y)}$ for each $x, y \in \mathcal{H}$;

(3) *positive semidefinite* if b is Hermitian and for each $x \in \mathcal{H}$, $b(x, x) \geq 0$;

(4) *positive definite* if b is positive semidefinite and $b(x, x) = 0$ implies $x = 0$. In this case, b is called an *inner product*.

(b) If $\mathcal{H}$ is equipped with a positive definite sesquilinear form b, then we write

$$\langle x, y \rangle \equiv b(x, y)$$

and say that $\mathcal{H}$ is an *inner product space*.

(c) Two elements $x, y \in \mathcal{H}$ are called *orthogonal* if $\langle x, y \rangle = 0$, written $x \perp y$ or $y \perp x$. For a subset $E \subset \mathcal{H}$, we write

$$E^{\perp} = \{y \in \mathcal{H} : \text{ for all } x \in E, \text{ we have } \langle y, x \rangle = 0\}\,.$$

In the real case, we simply ignore the complex conjugate and apply these definitions verbatim, as for any $\lambda \in \mathbb{R}$, $\overline{\lambda} = \lambda$.

1.2 Exercise. Note that $\perp$ is a symmetric relation on $\mathcal{H}$. From which property in our definitions does this symmetry originate?

We revisit two fundamental quantitative results for Hermitian sesquilinear forms.

1.3 Lemma. *Let* b *be a Hermitian sesquilinear form on a complex vector space* $\mathcal{V}$*.*

(a) If b *is positive semidefinite, then for* $x, y \in \mathcal{V}$*, the* Cauchy-Schwarz Inequality *holds,*

$$|b(x,y)|^2 \leq b(x,x)b(y,y)\,.$$

(b) For $x, y \in \mathcal{V}$*, the* Polarization Identity

$$b(x,y) = \frac{1}{4}(b(x+y,x+y) - b(x-y,x-y) + ib(x+iy,x+iy) - ib(x-iy,x-iy))$$

is satisfied.

We note that the form of the polarization identity depends on the convention that the inner product is conjugate in the second entry. The quantitative insights related to Hermitian sesquilinear forms help characterize inner product spaces in terms of the associated norm.

1.4 Theorem (Jordan, v. Neumann)**.** *If* $\mathcal{H}$ *is a complex inner product space, then* $\|x\| \equiv (\langle x,x\rangle)^{1/2}$ *defines a norm on* $\mathcal{H}$ *which satisfies that for* $x, y \in \mathcal{H}$*, the* parallelogram law *holds,*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2\,.$$

Conversely, if $\mathcal{H}$ *is a normed space for which the parallelogram law holds, then*

$$b(x,y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

defines an inner product on $\mathcal{H}$*.*

Proof. Assuming $\mathcal{H}$ is an inner product space, then by the Cauchy-Schwarz inequality,

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle \\ &= \|x\|^2 + 2\mathrm{Re}[\langle x,y\rangle] + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2\,.\end{aligned}$$

Now taking the square root on both sides gives the triangle inequality. The other properties of the norm are left as an exercise.

Conversely, if a norm satisfies the parallelogram identity, then

$$\|x+y\|^2 - \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2 = 4(\|x\|^2 + 4\|y\|^2),$$

so we can deduce that this quantity is zero if and only if $x = y = 0$.

We use this property together with other, relatively elementary steps to show that b is Hermitian and positive definite. The suggested steps are

Step 1. Show for each $x \in V$, $b(x, x) \geq 0$ and $b(x, x) = 0$ is equivalent to $x = 0$;

Step 2. for each $x \in \mathcal{V}$, $\lambda \in \mathbb{C}$, $b(\lambda x, \lambda x) = |\lambda|^2 b(x, x)$;

Step 3. for each $x, y \in \mathcal{V}$, $b(x, y) = \overline{b(y, x)}$,

Step 4. for each $x, y, z \in \mathcal{V}$, $b(x, y + z) = 2b(\frac{x}{2}, y) + 2b(\frac{z}{2}, z)$;

Step 5. for each $x, y \in \mathcal{V}$, $q \in \mathbb{Q}$, $b(qx, y) = qb(x, y)$;

Step 6. for each $qx, y \in \mathcal{V}$, $\lambda \in \mathbb{C}$, $b(\lambda x, y) = \lambda b(x, y)$.

□

The norm induces a metric topology on any inner product space. If the inner product space is complete with respect to this metric, so all Cauchy sequences converge, then we have the nicest possible setting.

1.5 Definition. An inner product space $\mathcal{H}$ that is complete with respect to the topology induced by the norm associated with the inner product is called a *Hilbert space*.