Lecture Notes from August 25, 2022

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Last time

- From inner product spaces to Hilbert spaces,
- orthogonality, orthogonal complements,
- Chauchy-Schwarz inequality and parallelogram law,
- polarization identity
- Jordan-von-Neumann Theorem.

Warm up:

1.6 Question. If $\mathcal{H}$ is a Hilbert space and y a fixed vector, why is the linear functional $\Lambda_y : x \mapsto \langle x, y\rangle$ a continuous map?

This is because of the Cauchy-Schwarz inequality, $|\langle x-z, y\rangle| \le \|y\|\|x-z\|$, so Λ_y is in fact Lipschitz continuous with Lipschitz constant $\|y\|$. This constant is also the operator norm of Λ_y, because $\sup_{x:\|x\|\le 1} \|\Lambda_y x\| = \|y\|$.

1.7 Question. If E is a subset of a Hilbert space, why is $E^\perp$ closed?

To see this, we write

$$E^\perp = \cap_{y\in E}\{x \in \mathcal{H} : \langle x, y\rangle = 0\}$$

and note that each set $\{x \in \mathcal{H} : \langle x, y\rangle = 0\} = \Lambda_y^{-1}(\{0\})$ is closed because it is the inverse image of a closed set under a continuous map. In fact, from the kernel of Λ_y being a subspace, we see $E^\perp$ is the intersection of closed subspaced, thus itself a closed subspace.

We recall that completeness is a key property of Hilbert spaces. Fortunately, one can always pass from an inner product space to a possibly larger Hilbert space.

1.8 Theorem. *If $\mathcal{H}$ is an inner product space and $\widehat{\mathcal{H}}$ the (metric) completion of $\mathcal{H}$, then the inner product on $\mathcal{H}$ extends uniquely to an inner product on $\widehat{\mathcal{H}}$.*

To see this, one considers the extension of the associated norm on $\mathcal{H}$, which is uniformly continuous. By continuity of the extension, the resulting norm on $\widehat{\mathcal{H}}$ is uniquely determined and satisfies the parallelogram identity, hence belongs to an inner product. Using that $\mathcal{H}$ is dense in its completion, the continuity of the inner product on $\widehat{\mathcal{H}}$ shows that this is the unique continuous extension of the inner product from $\mathcal{H}$.

We consider examples of Hilbert spaces.

1.9 Examples. 1. The n-dimensional complex Euclidean space $\mathbb{C}^n$ is equipped with the inner product $\langle x, y\rangle = \sum_{j=1}^{n} x_j \overline{y_j}$ which turns it into a Hilbert space.

2. The space of complex square-summable sequences $\ell^2 \equiv \ell^2(\mathbb{N})$ is also a Hilbert space when the inner product is chosen as $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j \overline{y_j}$.

3. The space of continuous functions $C([a, b])$ on the interval from a to b with

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$$

is an inner product space with completion $L^2([a, b])$.

Next, we review the most fundamental results on orthogonality.

1.10 Theorem. *The orthogonal complement has the following properties:*

(a) If F is a closed subspace of a Hilbert space, then $\mathcal{H} = F \oplus F^{\perp}$, so $\mathcal{H}$ is the direct sum of the (closed) subspaces F and $F^{\perp}$.

(b) If $E \subset \mathcal{H}$ is a subset, then $(E^{\perp})^{\perp} = \overline{\mathrm{span} E}$. In particular, $E = (E^{\perp})^{\perp}$ if and only if E is a closed subspace.

Before proving the two parts of this theorem, we introduce a special linear map associated with closed subspaces.

1.11 Definition. Let $\mathcal{H}$ be a Hilbert space and F a closed subspace, then there is a bounded linear map $P : \mathcal{H} \to \mathcal{H}$ such that for each $x \in \mathcal{H}$, $Px \in F$ and $\|x - Px\| \leq \|x - y\|$ for each $y \in F$.

We call P the orthogonal projection onto F, which becomes clear when considering the following geometric property.

1.12 Proposition. *If P is the orthogonal projection associated with a closed subspace F in a Hilbert space, then for each $x \in \mathcal{H}$, $y \in F$,*

$$\langle x - Px, y\rangle = 0\,.$$

Proof. Taking squares, for any $y \in F$ and $t \in \mathbb{R}$, we have

$$\|x - Px\|^2 \leq \|x - Px + ty\|^2\,.$$

So at $t = 0$ the right-hand side achieves its minimum and by this being a real quadratic polynomial, the derivative vanishes, so

$$2\mathrm{Re}[\langle x - Px, y\rangle] = 0\,.$$

Replacing y by iy and using sesqui-linearity of the inner product gives that the derivative with respect to t at $t = 0$ yields

$$2\mathrm{Im}[\langle x - Px, y\rangle] = 0\,.$$

We conclude $x - Px \perp y$. □

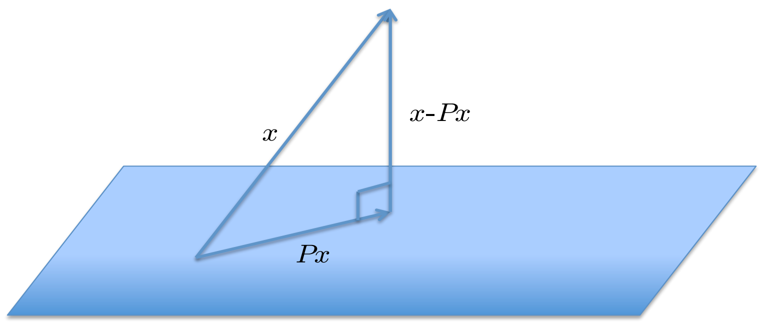

Figure 1: The relationship between the orthogonal projection of a vector x, $x - Px$, and the range of P (the subspace shaded in blue) is illustrated here. In particular, $x - Px \perp Px$.

This orthogonal relationship between $x - Px$ and F is sketched in a drawing in Figure 1.

1.13 Corollary. *By the orthogonality relation, we have Pythagoras* $\|x\|^2 = \|Px\|^2 + \|x - Px\|^2$.

Next, we prove the two outstanding parts of the theorem.

Proof of Theorem (a). Let $x \in \mathcal{H}$, then by the above,

$$x - Px \in F^{\perp}.$$

so $x = Px + (x - Px)$ and the two summands are from the spaces F and $F\perp$, hence $\mathcal{H} = F + F^{\perp}$.

In fact, this decomposition is unique. Assuming $x = y_1 + z_1 = y_2 + z_2$ with $y_1, y_2 \in F$ and $z_1, z_2 \in F^{\perp}$, then

$$z_1 - z_2 = y_2 - y_1$$

and the left hand side is a vector in $F^{\perp}$, the right hand side in F, and both sides are equal, so they must be $\{0\} = F \cap F^{\perp}$. We conclude $z_1 = z_2$ and $y_2 = y_1$, the claimed uniqueness. □

We continue with proving the second part of the theorem.

Proof of Theorem (b). Take $x \in E$. By definition, for each $y \in E^{\perp}$, $\langle x, y \rangle = 0$, so $x \in (E^{\perp})^{\perp}$ and we have shown $E \subset (E^{\perp})^{\perp}$.

What is left is the reverse inclusion. From $(E^\perp)^\perp$ being an orthogonal complement, it is a closed subspace (see warm-up exercise). This means we can retain the inclusion upon enlarging E to its closed linear span

$$\overline{\mathrm{span}(E)} \subset (E^\perp)^\perp .$$

Now considering $F = \overline{\mathrm{span}(E)}$ and any $x \in (E^\perp)^\perp$, there is a unique decomposition $x = y + z$ with $y \in F$ and $z \in F^\perp$. Taking the inner product of both sides of this identity with z gives

$$\langle x, z \rangle = \|z\|^2 .$$

From the inclusion $\mathrm{span}(E) \subset F$, we get the reverse inclusion of the orthogonal complements $F^\perp \subset (\mathrm{span}(E))^\perp$, so $z \in F^\perp$ is in the orthogonal complement of E, and by $x \in (E^\perp)^\perp$, we get the succession of identities $\langle x, z \rangle = 0$, $\|z\|^2 = 0$, $z = 0$, and finally $x = y$. Thus, $x \in F$, which proves the inclusion $(E^\perp)^\perp \subset \overline{\mathrm{span}(E)}$.

We conclude $(E^\perp)^\perp = \overline{\mathrm{span}(E)}$.

□