Dual Spaces, Riesz Representation Theorem, and Summability

Lecture Notes from August 30, 2022

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Last Time

- Direct sum
- Orthogonal projections and orthogonal spaces

Warm up

- Recall the corollary from last time: Suppose E is a subspace of a Hilbert space $\mathcal{H}$. Then E is closed if and only if $(E^{\perp})^{\perp} = E$. The proof for the preceding theorem was quite lengthy, but a shorter proof of the corollary can be found in Rudin: Since E is closed, the direct sums
$$\bar{E} \oplus \bar{E}^{\perp} = \mathcal{H}$$
and
$$\bar{E}^{\perp} \oplus (\bar{E}^{\perp})^{\perp} = \mathcal{H}$$
are unique. Comparing the two identities, we identify that $\bar{E}$ must be $(\bar{E}^{\perp})^{\perp}$.

- Recall the steps to prove completeness of $l^2 \equiv l^2(\mathbb{N})$:
  1. Consider any Cauchy sequence $(x^n)_{n\in\mathbb{N}}$ in l^2. Using norm inequality
  $$|x_j^n - x_j^m| \leq \|x^n - x^m\|,$$
  we get that, for each fixed j, $(x_j^n)_{n\in\mathbb{N}}$ forms a Cauchy sequence in $\mathbb{C}$, hence converges to $x_j^n \to x_j$ by completeness.
  2. Using that Cauchy sequences are bounded, show $(x_j)_{j\in\mathbb{N}} \in l^2(\mathbb{N})$.
  3. Show sequence $(x^n)_{n\in\mathbb{N}}$ actually converges to x with respect to the norm on l^2, which can be shown by using triangle inequality and $\sup$ argument.

Dual Spaces

Consider the space of bounded linear functionals on $\mathcal{H}$.

1.14 Definition. The *dual space* V' of a normed space V is given by all linear maps

$$\Lambda : V \to \mathbb{C}, \quad \sup_{\|x\| \leq 1} |\Lambda x| < \infty.$$

We equip V' with the norm

$$\|\Lambda\| = \sup_{\|x\| \leq 1} |\Lambda x|.$$

To see that $\|\Lambda\|$ is indeed a norm on $\mathcal{H}$, it is enough to show that it is positive definite, ie., we want to show that

$$\|\Lambda\| = 0 \iff \Lambda = 0.$$

Suppose that $\|\Lambda\| = 0$, or $\sup_{\|x\| \leq 1} |\Lambda x| = 0$, which implies $|\Lambda x| \leq 0$ for all $x \in \mathcal{H}$, so it must be true that $\Lambda x = 0$ for all x. Thus, $\Lambda = 0$.

Conversely, let $\Lambda = 0$, then $\sup_{\|x\| \leq 1} |\Lambda x| = 0 = \|\Lambda\|$. □

Riesz Representation Theorem

1.15 Theorem. *Let $\mathcal{H}$ be a complex Hilbert space. Then the map $\phi : \mathcal{H} \to \mathcal{H}'$ given by*

$$(\phi(x))\,(y) = \langle y, x \rangle$$

is a conjugate linear isometric bijection.

In particular, if Λ is a bounded linear functional on $\mathcal{H}$, then there is an $x \in \mathcal{H}$ so that for each $y \in \mathcal{H}$,

$$\Lambda y = \langle y, x \rangle,$$

and

$$\|\Lambda\| = \|x\|.$$

Proof. (This proof may be found in Rudin, Chapter 12, page 308. I typed it here for convenience and added a few notes for my own understanding).

We first want to show $\|\Lambda\| = \|x\|$ for $x \in \mathcal{H}$. By CauchySchwarz inequality, we have that

$$|\Lambda y| := |\langle y, x \rangle| \leq \|y\| \|x\|,$$

which implies $\|\Lambda\| \leq \|x\|$. Conversely, $\|x\|^2 = \langle x, x \rangle =: |\Lambda x| \leq \|\Lambda\| \|x\|$, so $\|x\| \leq \|\Lambda\|$.

Next, we want to show that every $\Lambda \in \mathcal{H}$ has the form $\Lambda y = \langle y, x \rangle$ for $y \in \mathcal{H}$. If $\Lambda = 0$, then we take $x = 0$. If Λ is non-zero, we let $\mathcal{N}(\Lambda) := \{y \in \mathcal{H} : \Lambda y = 0\}$ be the null space of Λ. Since $\mathcal{H} = \mathcal{N}(\Lambda) \oplus (\mathcal{N}(\Lambda))^\perp$, there exists a $z \in (\mathcal{N}(\Lambda))^\perp$, $z \neq 0$, and

$$(\Lambda y)z - (\Lambda z)y \in \mathcal{N}(\Lambda)$$

for all $y \in \mathcal{H}$, which implies

$$(\Lambda y)\langle z, z \rangle - (\Lambda z)\langle y, z \rangle = 0$$

$$\iff \Lambda y = \frac{1}{\langle z, z \rangle} \Lambda z \langle y, z \rangle = \langle y, \bar{\Lambda} z \frac{z}{\langle z, z \rangle} \rangle.$$

Hence, for any $y \in \mathcal{H}$, $\Lambda y = \langle y, x \rangle$ with $x = (\bar{\Lambda} z) \frac{z}{\langle z, z \rangle} \in \mathcal{H}$. □

Summability

In this section, we introduce a more general version of direct sum.

First, let us recall the definition of summability.

1.16 Definition. Let V be a normed space, and $(x_j)_{j\in J}$ a family of elements in V, so $x : J \to V, j \mapsto x_j$ a V-valued function.

Let $\mathcal{F}$ be the set of all finite subsets of J. Then $(x_j)_{j\in J}$ is called *summable* if there is a $y \in V$ such that for each $\epsilon > 0$, there is an $F_\epsilon \in \mathcal{F}$ such that for all $F \in \mathcal{F}$ with $F_\epsilon \subset F$,

$$\sum_{j\in F} x_j \in B_\epsilon(y) \equiv \{x \in V : \|x - y\| < \epsilon\}.$$

For example, suppose $V = \mathbb{R}$ and (x_j) are non-negative functions (not necessarily linear), given by $j \mapsto x_j \in \mathbb{R}^+$. If $(x_j)_{j\in J}$ are summable, then

$$\sup\{\sum_{k=1}^{n}(x_j)_k : \{j_1, j_2, ..., j_n\} \in J\} < \infty.$$

1.17 Remark. We note that this definition is, unlike the convergence of series, invariant reordering J with a bijection since any reordering would create another subset of J that is also finite and in $\mathcal{F}$.

We use the notion of summability to equip a family of Hilbert spaces with a new inner product.

1.18 Lemma. *Let $(\mathcal{H}_j)_{j\in J}$ be a family of Hilbert spaces and*

$$\mathcal{H} = \{(x_j)_{j\in J} \in \prod_{j\in J} \mathcal{H}_j, \sum_{j\in J} \|x_j\|^2 < \infty\}.$$

Then $\mathcal{H}$ is a Hilbert space with the inner product

$$\langle (x_j)_{j\in J}, (y_j)_{j\in J}\rangle = \sum_{j\in J} \langle x_j, y_j\rangle.$$

Proof. We first show that H is subspace of the vector space $\prod_{j\in J} \mathcal{H}_j$:

It is clear to see that $\mathcal{H}$ is closed under scalar multiplication. To see $\mathcal{H}$ is closed under addition, recall the parallelogram law

$$\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$$

for $a, b \in \mathcal{H}$. So for $x = (x_j)_{j\in J}, y = (y_j)_{j\in J} \in \mathcal{H}$,

$$\|x + y\|^2 = \sum_{j\in J} \|x_j + y_j\|^2 \leq 2\sum_{j\in J} \|x_j\|^2 + 2\sum_{j\in J} \|y_j\|^2 < \infty,$$

where the last inequality is due to the summability of x_j, y_j. Thus, $\mathcal{H}$ is a subspace of $\prod_{j\in J} \mathcal{H}_j$.

Next, for $x, y \in \mathcal{H}$, the polarization identity, $x \pm y \in \mathcal{H}$, and $x \pm iy \in \mathcal{H}$ gives

$$\langle x, y\rangle = \sum_{j\in J} \langle x_j, y_j\rangle.$$

To show that the above identity is an inner product, it is enough to show its positive definiteness. Let $x \in \mathcal{H}$. Suppose $\langle x, x \rangle = 0$, which implies $\sum_{j\in J} \langle x_j, y_j \rangle = 0$. Since $\mathcal{H}_j$ is a Hilbert space for each j, we have that, if $\langle x_j, x_j \rangle = 0$, then $x_j = 0$, so $x = 0$. Conversely, if $x = 0$, then $x_j = 0$ for all $j \in J$, and since $\mathcal{H}_j$ are Hilbert spaces, $\langle x_j, x_j \rangle = 0$ or $\sum_{j\in J} \langle x_j, x_j \rangle = 0 = \langle x, x \rangle$.

Finally, we need to show completeness (using the steps provided in the warm-up).

Let $(x^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}$. Then by norm inequality

$$\|x_j^n - x_j^k\| \leq \|x^n - x^k\|_{\mathcal{H}},$$

we have that, for fixed $j \in J$, $(x_j^n)_{n\in\mathbb{N}}$ is Cauchy, ie. $(x_j^n)_{j\in J}$ converges to $x_j^n \mapsto x_j \in \mathcal{H}_j$.

For each finite subset $F \subset J$:

$$\sum_{j\in F} \|x_j\|_{\mathcal{H}_j}^2 = \lim_{k\to\infty} \sum_{j\in F} \|x_j^k\|_{\mathcal{H}_j}^2 \leq \lim_{k\to\infty} \sum_{j\in J} \|x_j^k\|_{\mathcal{H}_j}^2 = \lim_{k\to\infty} \|x^k\|^2 < \infty,$$

with the last inequality using boundedness of Cauchy sequences. Hence, $(x_j)_{j\in J} \in \mathcal{H}$.

It remains to show that $x^n \to x$ as $n \to \infty$. Let $\epsilon > 0$ and choose $n_0 \in \mathbb{N}$ with $\|x^n - x^k\| < \epsilon$ for $n, k \geq n_0$. Then for each finite $F \in J$

$$\sum_{j\in F} \|x_j - x_j^n\|_{\mathcal{H}}^2 = \lim_{m\to\infty} \sum_{j\in F} \|x_j^m - x_j^n\|_{\mathcal{H}}^2 \leq \lim_{m\to\infty} \sum_{j\in J} \|x_j^m - x_j^n\|_{\mathcal{H}}^2 \leq \epsilon^2,$$

so $x^n \to x$, and $\mathcal{H}$ is complete. $\square$

The readers may notice that, if $\mathcal{H} = \mathbb{R}^n = \prod_n \mathbb{R}$, then the inner product in the above theorem is the same with the usual inner product in $\mathbb{R}^n$.