Lecture Notes from August 30, 2022

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Warm Up Exercises

Recall the following:

1.1.14 Definition. For any subset E of a Hilbert Space $\mathcal{H}$, $E^{\perp} := \{x \in \mathcal{H} : \forall y \in E, \langle x, y \rangle = 0\}$

1.1.15 Theorem. *If F is a closed subspace of a Hilbert Space $\mathcal{H}$, then $\mathcal{H} = F \oplus F^{\perp}$.*

1.1.16 Exercise. Suppose we have a closed subspace $\overline{V}$. Then $\overline{V}$ is a closed subspace of a Hilbert space $\mathcal{H}$, thus

$$\overline{V} \oplus \overline{V}^{\perp} = \mathcal{H}$$

However, $\overline{V}^{\perp}$ is also a closed subspace of a Hilbert Space, therefore

$$\overline{V}^{\perp} \oplus \overline{V}^{\perp^{\perp}} = \mathcal{H}$$

Comparing direct sums listed above, we have

$$(\overline{V}^{\perp})^{\perp} = \overline{V}$$

1.1.17 Exercise. Recall the steps to show $l^2 \equiv l^2(\mathbb{N})$ is complete.

(1) Consider a Cauchy sequence $(x^n)_{n \in \mathbb{N}}$ in l^2 The inequality $|x_j^n - x_j^m| \leq \|x^n - x^m\|$ gives us that for each j$\in \mathbb{N}$, $(x_j^n)_{n \in \mathbb{N}}$ forms a Cauchy sequence in $\mathbb{C}$, hence $x_j^n \to x_j$ by completeness

(2) By the boundedness of Cauchy sequences, show that $(x_j)_{j \in \mathbb{N}} \in l^2(\mathbb{N})$

(3) Show the sequence (x^n) converges to x w.r.t to the norm on l^2 using the triangle inequality and the sup argument.

The Dual

We begin a study of the space of bounded linear functionals on a Hilbert Space.

1.1.18 Definition. The *dual* V' of a normed vector space V is given by all linear maps $\lambda : V \to \mathbb{C}$ such that $sup_{\|x\| \leq 1} |\lambda x| < \infty$. We equip V' with the norm $\|\lambda\| = sup_{\|x\| \leq 1} |\lambda x|$

1.1.19 Remark. $\|\lambda\|$ is positive definite. if λ is the zero functional, then $\|\lambda\| = sup_{\|x\|\leq 1}|\lambda x| = 0$. In the other direction, if $\|\lambda\| = 0$ then $|\lambda x| = 0$ for each $x \leq 1$. We can then apply continuous norm preserving extensions to show $|\lambda x| = 0$ on the entire Hilbert space, and thus $\|\lambda\|$ is the zero functional. This shows both directions of the positive definite definition.

1.1.20 Lemma. *Let V be a normed vector space. Let $\lambda : V \to \mathbb{C}$ be a bounded linear functional on V. Then λ is continuous w.r.t the norm.*

Proof. Let M be an upper bound on λ. For any $\epsilon > 0$ there exists $\sigma < \frac{\epsilon}{M}$ such that

$$\|x - y\| < \sigma \implies |\lambda(x) - \lambda(y)| = |\lambda(x-y)| = \|x-y\||\lambda(\frac{x-y}{\|x-y\|}) < \sigma M < \frac{\epsilon}{M}M = \epsilon$$

□

1.1.21 Theorem (Riesz Representation Theorem)**.** *Let $\mathcal{H}$ be a complex Hilbert Space. Then the map $I : \mathcal{H} \to \mathcal{H}'$ defined by $(I(x))(y) = \langle y, x\rangle$ is a conjugate linear isometric bijection. In particular, if λ is a bounded linear functional, then there is a unique $x \in \mathcal{H}$ such that for each $y \in \mathcal{H}, \lambda y = \langle y, x\rangle$, and $\|\lambda\| = \|x\|$*

Proof. Let λ be a bounded linear functional on a Hilbert Space $\mathcal{H}$. If λ is the zero functional, then we have $\lambda(y) = \langle y, x\rangle = 0$ for $x = \vec{0} \in \mathcal{H}$. Now assume λ is not the zero functional. Then there exists $x \in \mathcal{H}$ such that $\lambda(x) \neq 0, x \notin ker\lambda$. λ is a bounded linear functional, hence by the lemma above λ is continuous. Therefore, $ker\lambda = \lambda^{-1}(\{0\})$ is closed by continuity. Thus, $\mathcal{H} = ker\lambda \oplus ker\lambda^{\perp}$, and $ker\lambda^{\perp}$ is also closed. Then by the Projection Theorem there exists a projection $P : \mathcal{H} \to ker\lambda^{\perp}$. Now define $u = P(x) \in ker\lambda^{\perp}$. Note that $P(x) \neq 0$ as $x \notin ker\lambda$. Then for each $y \in \mathcal{H}$ we have the following calculation:

$$\langle y, u\rangle = \langle y - \frac{\lambda(y)u}{\|u\|}, u\rangle + \langle\frac{\lambda(y)u}{\|x\|}, u\rangle$$

$u \in ker\lambda^{\perp}$, therefore $\frac{\lambda(y)u}{\|u\|} \in ker\lambda^{\perp}$. Hence, $y - \frac{\lambda(y)u}{\|u\|} \in ker\lambda$ which yields $\langle y - \frac{\lambda(y)u}{\|u\|}, u\rangle = 0$. This zeroes out a term in our above calculation, giving us

$$\langle y, u\rangle = \langle\frac{\lambda(y)u}{\|u\|}, u\rangle = \lambda(y)$$

Thus for any $y \in H$, $\lambda(y) = \langle y, x\rangle$. Furthermore,

$$\|\lambda\| = sup_{\|y\|\leq 1}|\lambda y| = sup_{\|y\|\leq 1}|\langle y, x\rangle| = |x|$$

Finally, we prove uniqueness: Suppose there exists $x, x' \in \mathcal{H}$ such that $\lambda y = \langle y, x\rangle$ and $\lambda y = \langle y, x'\rangle$ for each $y \in \mathcal{H}$. Then we have

$$0 = \lambda y - \lambda y = \langle y, x\rangle - \langle y, x'\rangle = \langle y, x - x'\rangle$$

for each $y \in \mathcal{H}$. Thus

$$\langle x - x', x - x'\rangle = \|x - x'\| = 0$$

and so x - x' = 0 by the positive definite property of the norm, which gives x = x'. □

Summability

Recall that an infinite sum in $\mathbb{R}$ converges to $y \in \mathbb{R}$ iff for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N : |S_n - y| \leq \epsilon$ where S_n are the the partial sums. However, what if the index of the sum was potentially uncountable and $\mathbb{R}$ is any normed vector space? This leads us to our formal definition of summability.

1.1.22 Definition. Let V be a normed space, and let $(x_j)_{j\in J}$ be a family of elements in V which can be represented by a V - valued function $x : J \to V, j \to x$. Let $\mathcal{F}$ be the collection of all finite subsets of J. Then $(x_j)_{j\in J}$ is called *summable* if there exists $y \in V$ such that for each $\epsilon > 0$ there exists $F_\epsilon \in \mathcal{F}$ where for all $F \in \mathcal{F}$ with $F_\epsilon \subset F$, $\sum_{n\in F} x_j \in \beta_\epsilon(y) \equiv \{x \in V : \|x - y\| < \epsilon\}$

1.1.23 Exercise. Show that if $(x_j)_{j\in J}$ is given by $j \to x_j \in \mathbb{R}^+ \equiv [0, \infty)$, such that $V = \mathbb{R}^+$ and if $(x_j)_{j\in J}$ is summable then $sup\{\sum_{k=1}^n x_{j_k} : \{j_1, ...j_n\} \subset J\} < \infty$

Proof. Suppose $u := \{\sum_{k=1}^n x_{j_k} : \{j_1, ...j_n\} \subset J\}$ is not bounded above in $\mathbb{R}^+$. Then for $y \in \mathbb{R}^+, \epsilon > 0$ there exists $\{j_1, ...j_n\} \subset J$ such that $\sum_{k=1}^n x_{j_k} - y \geq \epsilon$. However $(x_j)_{j\in J}$ is summable, hence there exists a $F := \{l_1, ..., l_m\} \subset J$ such that $|\sum_{j\in\mathcal{F}} x_j - y| < \epsilon$ for $F \subset \mathcal{F}$ where $\mathcal{F}$ is any finite subset of J. Consider $S := F \cup \{j_1, ..., j_n\}$. We have $|\sum_{j\in S} x_j - y| < \epsilon$ by the above statement. However, $|\sum_{j\in S} x_j - y| \geq |\sum_{k=1}^n x_{j_k} - y| \geq \epsilon$ as the absolute value function is monotone increasing on $\mathbb{R}^+$ and $\sum_{j\in S} x_j - y \geq \sum_{k=1}^n x_{j_k} - y \geq \epsilon > 0$. This contradicts summability, and thus u must be bounded above. Now u is bounded above in $\mathbb{R}^+$, hence there exists $sup\{\sum_{k=1}^n x_{j_k} : \{j_1, ...j_n\} \subset J\} < \infty$ □

1.1.24 Theorem. *Let $(\mathcal{H}_j)_{j\in J}$ be a family of Hilbert Spaces and $\mathcal{H} = \{(x_j)_{j\in J} \in \prod_{j\in J} H_j : \sum_{j\in J} \|x_j\|^2 < \infty\}$. If $(\|x_j\|^2)_{j\in J}$ forms a summable family in $\mathbb{R}$ then $\mathcal{H}$ is a Hilbert Space with inner-product $\langle (x_j)_{j\in J}, (y_i)_{i\in J}\rangle = \sum_{j\in J}\langle x_j, y_j\rangle$*

Proof. We first show that $\mathcal{H}$ is a subspace of the vector space $\prod_{j\in J} \mathcal{H}_j$. Closure under scalar multiplication is clear, and for $a, b \in \mathcal{H}_j$ we have by the parallelogram law $\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$. Then for $(x_j)_{j\in J} \in \mathcal{H}$ and $(y_j)_{j\in J} \in \mathcal{H}$

$$\sum_{j\in J} \|x_j + y_j\|^2 \leq 2\sum_{j\in J} \|x_j\|^2 + 2\sum_{j\in J} \|y_j\|^2 < \infty$$

Thus $x + y \in \mathcal{H}$, so $\mathcal{H}$ is vector space.

Now it will be verified that $\langle x, y\rangle = \sum_{j\in J}\langle x_j, y_j\rangle$ is positive definite, sesquilinear, Hermitian and thus an inner product.

(sesquilinear) For $x_1, x_2, y \in \mathcal{H}$ we have

$$\langle x_1 + x_2, y\rangle = \sum_{j\in J}\langle x_{1j} + x_{2j}, y_j\rangle = \sum_{j\in J}\langle x_{1j}, y_j\rangle + \sum_{j\in J}\langle x_{2j}, y_j\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$$

and also for $x, y, z \in H$, $a \in \mathbb{C}$

$$\langle x, ay+z\rangle = \sum_{j\in J}\langle x_j, ay_j+z_j\rangle = \sum_{j\in J}\overline{a}\langle x_j, y_j\rangle+\langle x_j, z_j\rangle = \overline{a}\sum_{j\in J}\langle x_j, y_j\rangle+\sum_{j\in J}\langle x_j, z_j\rangle = \overline{a}\langle x, y\rangle+\langle x, z\rangle$$

This shows that the sum is sesquilinear.

(Hermitian) Suppose $x, y \in \mathcal{H}$ Then we have

$$\langle x, y\rangle = \sum_{j\in J}\langle x_j, y_j\rangle = \sum_{j\in J}\overline{\langle y_j, x_j\rangle} = \overline{\sum_{j\in J}\langle x_j, y_j\rangle} = \overline{\langle x, y\rangle}$$

Thus our sum is Hermitian.

(Positive Definite) Suppose $x \in \mathcal{H}$. If $\langle x, x\rangle = 0$, then

$$\langle x, x\rangle = \sum_{j\in J}\langle x_j, x_j\rangle = \sum_{j\in J}\|x_j\|^2 = 0$$

hence for each $j \in J$, $\|x_j\| = 0$ and thus $x_j = 0$ by positive definite property of norms. Thus $x = \vec{0}$. Also, for each $x \in \mathcal{H}$ we have

$$\langle x, x\rangle = \sum_{j\in J}\langle x_j, x_j\rangle = \sum_{j\in J}\|x_j\|^2 \geq 0$$

Hence, $\langle x, y\rangle$ is positive semidefinite and $\langle x, x\rangle = 0 \implies x = \vec{0}$. Therefore $\langle x, y\rangle$ is positive definite. $\langle x, y\rangle = \sum_{j\in J}\langle x_j, y_j\rangle$ is positive definite, sesquilinear, Hermitian and thus an inner product, which allows us to use the polarization identity. For $x, y \in \mathcal{H}$, we have $x+iy, x-iy \in \mathcal{H}$, thus

$$\langle x, y\rangle = \sum_{j\in J}\langle x_j, y_j\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2) \in \mathcal{H}$$

Hence the sum $\sum_{j\in J}\langle x_j, y_j\rangle$ exists in $\mathcal{H}$ and is a well defined inner product on $\mathcal{H}$. It remains to prove completeness by borrowing from the proof of completeness of l^2. Let $(x^n)_{n\in\mathbb{N}}$ be Cauchy in $\mathcal{H}$. Then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n, m \geq N$, $\|x_j^n - x_j^m\| \leq \|x^n - x^m\| < \epsilon$ which gives us that for fixed $j \in J$, $x_j^n \to x_j \in \mathcal{H}_j$ by completeness of $\mathcal{H}_j$.

Let $x \in \mathcal{H}$ such that $x_j = \lim_{n\to\infty} x_j^n$. Then for each finite subset $F \subset J$,

$$\sum_{j\in F}\|x_j\|_{\mathcal{H}_j}{}^2 = \lim_{n\to\infty}\sum_{j\in F}\|x_j^n\|_{\mathcal{H}_j}{}^2 \leq \lim_{n\to\infty}\sum_{j\in J}\|x_j^n\|_{\mathcal{H}_j}^2 = \lim_{n\to\infty}\|x^n\|^2 < \infty$$

by boundedness of Cauchy sequences, hence $(x_j)_{j\in J} \in \mathcal{H}$

It remains to show $\lim_{n\to\infty}(x^n) = x$.
Let $\epsilon > 0$, and choose $N_o \in \mathbb{N}$ such that for $n, m \geq N_o$ we have $\|x^n - x^m\| \leq \epsilon$. Then for each finite $F \subset J$,

$$\sum_{j\in F}\|x_j - x_j^n\|_{\mathcal{H}_j}^2 = \lim_{m\to\infty}\sum_{j\in F}\|x_j^m - x_j^n\|_{\mathcal{H}_j}^2 \leq \lim_{m\to\infty}\|x^m - x^n\|_{\mathcal{H}}^2 \leq \epsilon^2$$

thus we have $x^n \to x \in \mathcal{H}$ and so $\mathcal{H}$ is a complete inner-product space. This completes the proof that $\mathcal{H}$ is a Hilbert Space.

□