Lecture Notes from September 1, 2022

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Last time

- Dual Spaces
- Riesz Representation Theorem
- Summability

Warm up:

1.25 Question. Why does $x \in l^2([0,1])$ have at most countably many non-zero elements?

Consider a positive sequence of real numbers converging to zero monotonically. E.g. $\{1/n\}$. Let $\mathcal{J}_n = \{j \in [0,1] : |x_j|^2 > 1/n\}$. Then,

$$\bigcup_{n\in\mathbb{N}} \mathcal{J}_n = \{j \in [0,1] : |x_j|^2 > 0\}.$$

By the definition of $\|\cdot\|$ in $l^2([0,1])$, for any $n \in \mathbb{N}$,

$$\begin{aligned} \|x\|^2 &\geq \sup_{\substack{\mathcal{F}\subset\mathcal{J}_n \\ \mathcal{F} \text{ finite}}} \sum_{j\in\mathcal{F}} |x|^2 \\ &\geq \sup_{\substack{\mathcal{F}\subset\mathcal{J}_n \\ \mathcal{F} \text{ finite}}} \sum_{j\in\mathcal{F}} \frac{1}{n} \\ &= \sup_{\substack{\mathcal{F}\subset\mathcal{J}_n \\ \mathcal{F} \text{ finite}}} \frac{|F|}{n} = \frac{|\mathcal{J}_n|}{n}. \end{aligned}$$

Because $x \in l^2([0,1])$, $\|x\|^2 < \infty$, so $\mathcal{J}_n$ must be finite. Since each $\mathcal{J}_n$ is finite, $\cup_{n\in\mathbb{N}}\mathcal{J}_n$ is countable. Hence, there are at most countably many non-zero elements of x.

1.26 Definition. For a family of Hilbert Spaces $(\mathcal{H})_{j\in\mathcal{J}}$ we define

$$\mathcal{H} = \widehat{\bigoplus_{j\in\mathcal{J}}} = \{(x_j)_{j\in\mathcal{J}} \in \prod_{j\in\mathcal{J}} \mathcal{H}_j : \sum_{j\in\mathcal{J}} \|x_j\|^2_{\mathcal{H}_j} < \infty\}\}$$

to be the direct sum of the Hilbert spaces $(\mathcal{H}_j)_{j\in\mathcal{J}}$

1.27 Remark. The sum $\sum_{j\in\mathcal{J}}\mathcal{H}_j$ of $\mathcal{H}_j$'s is given by finite linear combinations of

$$\mathcal{H}_j \cong \{(x_k)_{k\in\mathcal{J}} : \text{ for each } k \neq j, x_k = 0\} \subset \mathcal{H}.$$

This is not the same as $\mathcal{H}$, but $\sum_{j\in\mathcal{J}}\mathcal{H}_j$ is a direct sum and it is dense in $\mathcal{H}$. We observe that the summability of $(||x_j||^2)_{j\in\mathcal{J}}$ implies that only countably many x_j are non-zero.

Proof. Let $x \in \mathcal{H}_k$ and $y \in \mathcal{H}_l$ for $k \neq l$. Then, by Lemma 1.18,

$$\langle x, y\rangle_{\mathcal{H}} = \sum_j \langle x_j, y_j\rangle_{\mathcal{H}_j} = \langle x_k, 0\rangle_{\mathcal{H}_k} + \langle 0, y_l\rangle_{\mathcal{H}_l} = 0$$

Thus, $\mathcal{H}_k \perp \mathcal{H}_l$, so $\sum \mathcal{H}_j$ is a direct sum. (Addition and scalar multiplication of elements may be evaluated componentwise.)

Let $\phi_j : \mathcal{H}_j \to \sum \mathcal{H}_j \subset \mathcal{H}$ be the inclusion map and suppose $x \in \mathcal{H}$. Then, $x = (x_j)$ and $\sum_{j\in\mathcal{J}} ||x_j||^2_{\mathcal{H}_j} < \infty$. By the warm-up, only countably many x_j are nonzero. WLOG order them so $x = (x_i)_{i\in\mathbb{N}}$, where we simply are ignoring the entries where $x_j = 0$. We will show $x = \sum_{i=1}^{\infty} \phi_i(x_i)$. For a finite number of terms,

$$\begin{aligned}
||x - \sum_{i=1}^{n} \phi(x_i)||^2 &= \langle x - \sum_{i=1}^{n} \phi(x_i), x - \sum_{i=1}^{n} \phi(x_i)\rangle \\
&= \sum_{j\in\mathcal{J}} \langle (x - \sum_{i=1}^{n} \phi(x_i))_j, (x - \sum_{i=1}^{n} \phi(x_i))_j\rangle_{\mathcal{H}_j} \text{ by Lemma 1.18} \\
&= \sum_{i=n+1}^{\infty} \langle x_i, x_i\rangle_{\mathcal{H}_i} \text{ Recall } x_j = 0 \text{ for all but countable } j.
\end{aligned}$$

Thus, $\lim_{n\to\infty} ||x - \sum_{i=1}^{n} \phi(x_i)||^2 = \sum_{i=n+1}^{\infty} \langle x_i, x_i\rangle_{\mathcal{H}_i} = 0$. Therefore, $\overline{\sum \mathcal{H}_j} = \mathcal{H}$. □

1.28 Corollary. *Let $\mathcal{J}$ be a set and $\mathbb{C}^{\mathcal{J}}$ contain each function $x : \mathcal{J} \to \mathbb{C}$, $j \to x_j$. Then,*

$$l^2(\mathcal{J}) = \{x \in \mathbb{C}^{\mathcal{J}} : \sum_{j\in\mathcal{J}} |x_j|^2 < \infty\}$$

is a Hilbert space with inner product $\langle x, y\rangle = \sum_{j\in\mathcal{J}} x_j\overline{y_j}$ and norm $||x|| = (\sum_{j\in\mathcal{J}} |x_j|^2)^{1/2}$.

This is a special case of Lemma 1.18 from August 30.

1.29 Theorem. *Let $\mathcal{J}$, $\mathbb{C}^{\mathcal{J}}$ be as defined above. Then,*

$$l^1(\mathcal{J}) = \{x \in \mathbb{C}^{\mathcal{J}} : \sum_{j\in\mathcal{J}} |x_j| < \infty\}$$

is a Banach space with norm $||x||_1 = \sum_{j\in\mathcal{J}} |x_j|$.

Proof. Follows the same strategy as the proof for the Hilbert space case. □

1.30 Remark. If $\mathcal{J} = \{1, ..., n\}$ then $l^2(\mathcal{J}) \cong \mathbb{C}^{\mathcal{J}}$ and if $\mathcal{J} = \mathbb{N}$ then $l^2(\mathcal{J}) \cong l^2$.

Towards *orthonormalbasis*.

1.31 Definition. A family $(x_j)_{j\in\mathcal{J}}$ in an inner product space $\mathcal{H}$ is an orthogonal family if $\forall j, i \in \mathcal{J}$ $i \neq j \implies x_j \perp x_i$. Furthermore, if $||x_j|| = 1$ $\forall j \in \mathcal{J}$ then it is an orthonormal family.

1.32 Lemma. *An orthogonal family, $(x_j)_{j\in\mathcal{J}}$, in an inner product space $\mathcal{H}$ is summable if and only if $(||x_j||^2)_{j\in\mathcal{J}}$ is summable in $\mathbb{R}$. In this case, $||\sum_{j\in\mathcal{J}} x_j||^2 = \sum_{j\in\mathcal{J}} ||x_j||^2$ and $\{j : x_j \neq 0\}$ is countable.*

Recall the definition of summability of (x_j): $\exists x \in \mathcal{H}$ such that $\forall \epsilon > 0$ $\exists F_\epsilon$ finite subset of $\mathcal{J}$ such that if $F_\epsilon \subset F$, $\sum_F x_j \in B_\epsilon(x)$.

Proof. Suppose $F \subset \mathcal{J}$ is finite. Then, the orthogonality of (x_j) implies $||\sum_F x_j||^2 = \sum_F ||x_j||^2$ by repeated application of Pythagoras.
$(\implies)$ Suppose (x_j) is summable and $F \subset \mathcal{J}$ is finite. Then, for $\epsilon = 1$, $||\sum_F x_j - x|| < 1$. Hence, $||\sum_F x_j|| < ||x|| + 1 \implies \sum_F ||x_j||^2 = ||\sum_F x_j||^2 < (||x|| + 1)^2$.
Therefore, $\sup_F \sum_F ||x_j||^2 \leq (||x|| + 1)^2 < \infty$, so $(||x_j||^2)_{j\in\mathcal{J}}$ is summable.

$(\impliedby)$ Suppose $(||x_j||^2)_{j\in\mathcal{J}}$ is summable. Then, by the warm up, only countably many x_j are nonzero and can be ordered $(x_i)_{i\in\mathbb{N}}$. Let $x = \sum_{i=1}^{\infty} x_i$ and $S_n = \sum_{i=1}^{n} x_i$. Thus, for $m < n \in \mathbb{N}$, $||S_n - S_m||^2 = ||\sum_{i=m+1}^{n} x_i||^2 = \sum_{i=m+1}^{n} ||x_i||^2$ by orthogonality. By summability, (S_n) is Cauchy and hence converges in $\mathcal{H}$ to x. To satisfy the definition of summability, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $||x - S_N|| < \epsilon$. Therefore, we can choose $F_\epsilon = (x_i)_{i\leq N}$. $\square$

The summability properties give consequences for orthonormal families.

1.33 Theorem. *Let $(x_j)_{j\in\mathcal{J}}$ be an orthonormal family in a Hilbert space $\mathcal{H}$. Then*

1. *for each $x \in \mathcal{H}$ (i.e. we fix an x), $\sum_{j\in\mathcal{J}} |\langle x, x_j\rangle|^2 \leq ||x||^2$ (Bessel's Inequality)*

2. *for each $x \in \mathcal{H}$, $x = \sum_{j\in\mathcal{J}} \langle x, x_j\rangle x_j$ iff $\sum_{j\in\mathcal{J}} |\langle x, x_j\rangle|^2 = ||x||^2$ (Parseval's Identity).*

Proof. 1) Let $x \in \mathcal{H}$. Let $F \subset \mathcal{J}$ be a finite set and let $V = span(\{(x_j) : j \in F\})$ and P be the orthogonal projection onto V. Then, by Theorem 1.12 $||Px||^2 \leq ||x||^2$. Since $Px \in V$, $Px = \sum_F \langle Px, x_j\rangle x_j = \sum_F \langle x, x_j\rangle x_j$ (by 1.12 again). Thus, $||\sum_F \langle x, x_j\rangle x_j||^2 = ||Px||^2 \leq ||x||^2$. Since this inequality is true for all finite F, $\sup_F \sum_F |\langle x, x_j\rangle|^2 \leq ||x||^2$. Therefore, by the definition of summable, $\sum_{\mathcal{J}} |\langle x, x_j\rangle|^2 \leq ||x||^2$.

2) Let $x \in \mathcal{H}$. Then, by (i), $(|\langle x, x_j\rangle|_j^2)$ is summable and hence has only countably many

nonzero terms. Reorder with $(x_i)_{i\in\mathbb{N}}$. Thus, using the orthonormality of (x_i),

$$\begin{aligned}
0 \leq ||x-\sum_{i=1}^{n}\langle x,x_i\rangle x_i||^2 &= \langle x-\sum_{i=1}^{n}\langle x,x_i\rangle x_i, x-\sum_{i=1}^{n}\langle x,x_i\rangle x_i\rangle \\
&= \langle x,x\rangle - \langle x-\sum_{i=1}^{n}\langle x,x_i\rangle x_i, x\rangle - \langle x, x-\sum_{i=1}^{n}\langle x,x_i\rangle x_i\rangle + \langle\sum_{i=1}^{n}\langle x,x_i\rangle x_i, \sum_{i=1}^{n}\langle x,x_i\rangle x_i\rangle \\
&= \langle x,x\rangle - 2Re(\langle x, x-\sum_{i=1}^{n}\langle x,x_i\rangle x_i\rangle) + \sum_{i=1}^{n}\langle x,x_i\rangle\overline{\langle x,x_i\rangle}\langle x_i,x_i\rangle \\
&= ||x||^2 - 2Re(\sum_{i=1}^{n}\overline{\langle x,x\rangle}\langle x,x_i\rangle + \sum_{i=1}^{n}|\langle x,x_i\rangle|^2 \\
&= ||x||^2 - \sum_{i=1}^{n}|\langle x,x_i\rangle|^2
\end{aligned}$$

Hence,

$$||x-\sum_{i=1}^{n}\langle x,x_i\rangle x_i||^2 = ||x||^2 - \sum_{i=1}^{n}|\langle x,x_i\rangle|^2.$$

Both sides of the equality are sequences of real numbers so LHS converges to zero if and only if the RHS converges to zero. The LHS converging to zero is equivalent to $\sum_{i=1}^{\infty}\langle x,x_i\rangle x_i = x \implies x = \sum_{j\in\mathcal{J}}\langle x,x_j\rangle x_j$ since the $(\langle x,x_i\rangle)$'s were the only nonzero elements. □

1.34 Definition. A subset B of a Hilbert space $\mathcal{H}$ is called *total* if $\overline{\mathrm{span}(B)} = \mathcal{H}$. A family, $(x_j)_{j\in\mathcal{J}}$, is called an *orthonormalbasis* (ONB) if $\cup_{j\in\mathcal{J}}\{x_j\}$ is total and $(x_j)_{j\in\mathcal{J}}$ is orthonormal.

1.35 Theorem. *If $(x_j)_{j\in\mathcal{J}}$ is an orthonormal family in a Hilbert space $\mathcal{H}$, then the following are equivalent:*

1. $(x_j)_{j\in\mathcal{J}}$ *is an ONB*
2. $(x_j)_{j\in\mathcal{J}}$ *is a maximal orthonormal family*
3. *If* $\langle x,x_j\rangle = 0\ \forall j\in\mathcal{J}$ *then* $x=0$
4. $\forall x\in\mathcal{H}$, $x=\sum_{j\in\mathcal{J}}\langle x,x_j\rangle x_j$
5. $\forall x,y\in\mathcal{H}$, $\langle x,y\rangle = \sum_{j\in\mathcal{J}}\langle x,x_j\rangle\langle x_j,y\rangle = \sum_{j\in\mathcal{J}}\langle x,x_j\rangle\overline{\langle y,x_j\rangle}$.
6. $\forall x\in\mathcal{H}$, $\sum_{j\in\mathcal{J}}|\langle x,x_j\rangle|^2 = ||x||^2$.

Proof. ($4 \iff 6$) Immediate by the previous theorem.
($1 \implies 2$) Suppose by contrapositive that (x_j) is not a maximal orthonormal family. Then, $\exists z\in\mathcal{H}$ such that $||z||=1$ and $z\perp x_j$ for all $j\in\mathcal{J}$. Thus, since $\mathcal{H} = \overline{\mathrm{span}(x_j)}\bigoplus\overline{\mathrm{span}(x_j)}^{\perp}$, and $\overline{\mathrm{span}(x_j)}^{\perp}$ is nonempty, $\overline{\mathrm{span}(x_j)}\neq\mathcal{H}$ so (x_j) is not an ONB.
($2 \implies 3$) Suppose by way of contradiction that $x\in\mathcal{H}$, $\langle x,x_j\rangle = 0$ for all $j\in\mathcal{J}$ and $x\neq 0$.

Then, $x/||x||$ has norm 1 and is also perpendicular to all x_j where $j \in \mathcal{J}$. This contradicts the maximality of (x_j).
$(1 \implies 4)$ We can use (3). By the previous theorem $\sum_{\mathcal{J}} |\langle x, x_j\rangle|^2 \leq ||x||^2$ and is thus summable, so at most a countable number of terms are nonzero. Since the order of the sum does not matter, WLOG we may write $\sum_{i=1}^{\infty} |\langle x, x_{j_i}\rangle|^2 \leq ||x||^2$. Let $S_n = \sum_{i=1}^{n} \langle x, x_{j_i}\rangle x_{j_i}$. Then, for $m < n$ $||S_n - S_m||^2 = ||\sum_{m+1}^{n} \langle x, x_{j_i}\rangle x_{j_i}||^2 = \sum_{m+1}^{n} |\langle x, x_{j_i}\rangle|^2$. Since the sum converges, this is a Cauchy sequence. Thus, $\exists z \in \mathcal{H}$ such that $z = \sum_{i=1}^{\infty} \langle x, x_{j_i}\rangle x_{j_i}$. Claim: $z - x \perp x_j$ for all $j \in J$. For the j's already discarded, $x \perp x_j$ and since those x_j's do not appear in the sum, so $z \perp x_j$. Also, for $k \in \mathbb{N}$,

$$\langle z - x, x_{j_k}\rangle = \langle \sum_{i=1}^{\infty} \langle x, x_{j_i}\rangle x_{j_i} - x, x_{j_k}\rangle = \langle x, x_{j_k}\rangle \langle x_{j_k}, x_{j_k}\rangle - \langle x, x_{j_k}\rangle = 0.$$

Thus, by 3, $z - x = 0$ so $x = z = \sum_{i=1}^{\infty} \langle x, x_{j_i}\rangle x_{j_i} = \sum_{j \in \mathcal{J}} \langle x, x_j\rangle x_j$.

So far we have proved 1 implies 2,3, and 4.

$(4 \implies 3)$ If we assume 4 and $\langle x, x_j\rangle = 0 \ \forall j \in \mathcal{J}$ then $x = \sum_{\mathcal{J}} 0 = 0$.
$(3 \implies 2)$ If we assume 3 then there can be no other norm 1 element to add to (x_j) that would be orthogonal to all currently given x_j elements. Hence (x_j) is a maximal orthonormal family.
$(2 \implies 1)$ By contrapositive suppose (x_j) is not an ONB for $\mathcal{H}$. Then, $\exists y \in \mathcal{H} - \overline{\text{span}(x_j)}$. Thus, $\mathcal{H} = \overline{\text{span}(x_j)} \bigoplus \overline{\text{span}(x_j)}^{\perp}$, so $\overline{\text{span}(x_j)}^{\perp}$ is nonempty. Hence, $\exists z \in \overline{\text{span}(x_j)}^{\perp}$ and $z/||z||$ has norm 1 and is perpendicular to $x_j \ \forall j \in \mathcal{J}$.

So far we have proved 1,2,3,4,6 are all equivalent.

$(4 \implies 5)$ If we assume 4 then, for $x, y \in \mathcal{H}$, $x = \sum_{\mathcal{J}} \langle x, x_j\rangle x_j$, which is summable, so as before, $x = \lim_{n\to\infty} \sum_{i=1}^{n} \langle x, x_i\rangle x_i$. Similarly for y. Thus, by the continuity of the inner product,

$$\begin{aligned}
\langle x, y\rangle &= \langle \lim_{n\to\infty} \sum_{i=1}^{n} \langle x, x_i\rangle x_i, \lim_{m\to\infty} \sum_{k=1}^{m} \langle y, x_k\rangle x_k\rangle \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \langle \sum_{i=1}^{m} \langle x, x_i\rangle x_i, \sum_{k=1}^{n} \langle y, x_k\rangle x_k\rangle \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{n} \sum_{k=1}^{m} \langle \langle x, x_i\rangle x_i, \langle y, x_k\rangle x_k\rangle \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{n} \sum_{k=1}^{m} \langle x, x_i\rangle \overline{\langle y, x_k\rangle} \langle x_i, x_k\rangle \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \langle x, x_i\rangle \overline{\langle y, x_i\rangle} \text{ by orthonormality} \\
&= \sum_{\mathcal{J}} \langle x, x_j\rangle \overline{\langle y, x_j\rangle} = \sum_{\mathcal{J}} \langle x, x_j\rangle \langle x_j, y\rangle.
\end{aligned}$$

($5 \implies 6$) Let $x \in \mathcal{H}$. Choosing $y = x$ in 5 gives

$$||x||^2 = \langle x, x\rangle = \sum_{\mathcal{J}} \langle x, x_j\rangle \overline{\langle x, x_j\rangle} = \sum_{\mathcal{J}} |\langle x, x_j\rangle|^2.$$

□

1.36 Corollary. *From the above, we see that if $(x_j)_{j\in\mathcal{J}}$ is an ONB then the map $x \mapsto (\langle x, x_j\rangle)_{j\in\mathcal{J}}$ is an isometric isomorphism from $\mathcal{H}$ to $l^2(\mathcal{J})$.*

Proof. Let $\Phi : \mathcal{H} \to l^2(\mathcal{J})$ be defined by $\Phi(x) = (\langle x, x_j\rangle)_{\mathcal{J}}$.
By 6, $||x||^2 = \sum_{\mathcal{J}} |\langle x, x_j\rangle|^2 = ||\Phi(x)||^2$. Thus, Φ is an isometry.
Suppose $x, y \in \mathcal{H}$ and $\Phi(x) = \Phi(y)$. Then, $\langle x, x_j\rangle = \langle y, x_j\rangle$ for all $j \in \mathcal{J}$. Thus, by 4 $x = \sum_{\mathcal{J}}\langle x, x_j\rangle = \sum_{\mathcal{J}}\langle y, x_j\rangle = y$. Hence, Φ is injective.
Suppose, $(a_j) \in l^2(\mathcal{J})$. Then, $\sum_{\mathcal{J}} |a_j|^2$ is summable in $\mathbb{R}$ so $\sum_{\mathcal{J}} a_j x_j$ is summable in $\mathcal{H}$. Let $x = \sum_{\mathcal{J}} a_j x_j$. Then, $\Phi(x) = (a_j)$ so Φ is surjective.
Let $a, b \in \mathbb{C}$ and $x, y \in \mathcal{H}$. Then,

$$\Phi(ax+y) = (\langle ax+y, x_j\rangle)_{\mathcal{J}} = (a\langle x, x_j\rangle + \langle y, x_j\rangle)_{\mathcal{J}} = a(\langle x, x_j\rangle)_{\mathcal{J}} + (\langle y, x_j\rangle)_{\mathcal{J}} = a\Phi(x) + \Phi(y).$$

Thus, Φ is linear.
By 5, we have that for $x, y \in \mathcal{H}$,

$$\langle \Phi(x), \Phi(y)\rangle = \langle (\langle x, x_j\rangle)_{\mathcal{J}}, (\langle x, x_j\rangle)_{\mathcal{J}}\rangle = \sum_{j\in\mathcal{J}} \langle x, x_j\rangle \overline{\langle y, x_j\rangle} = \langle x, y\rangle.$$

Therefore, Φ is an isometric isomorphism. □

Warm down:

1.37 Question. What is an ONB for $l^2([0, 1])$?

Choosing

$$x_j(r) = \begin{cases} 1 & r = j \\ 0 & r \neq j \end{cases}$$

gives an ONB for $l^2([0, 1])$.