Lecture Notes from September 6, 2022

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Last Time

- Dual Spaces
- Riesz Representation Theorem
- Summability
- Orthonormal bases

Warm up:

1.38 Question. Describe the dual space of $l^2([0,1])$.

From Riesz Rep, we know that every element in $(l^2([0,1]))'$ is a bounded linear functional $\Lambda : l^2([0,1]) \to \mathbb{C}$ given by

$$\Lambda x = \langle x, y \rangle = \sum_{j \in [0,1]} x_j \overline{y_j}$$

for a unique $y \in l^2([0,1])$, where from a previous lecture we know that $y_j \neq 0$ for at most countably many j.

Operators on Hilbert Spaces

We prepare for the discussion of spectral theory. The main ingredient is the map from an operator A to its adjoint A^*. We recall the equivalence between continuity and boundedness.

1.39 Theorem. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ a linear map. Then A is continuous if and only if $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$.*

1.40 Remark. For linear maps, continuity of a linear map $\Leftrightarrow$ continuity at zero, since any open neighborhood of a point x can be "linearly" translated to an open neighborhood of 0 and vice versa.

1.41 Remark. In this case, $\|A\|$ is the Lipschitz constant of the (Lipschitz) continuous map A. This motivates calling A bounded, because $A(B_1(0))$ is.

1.42 Definition. We write $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ for the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, and $\mathcal{B}(\mathcal{H})$ for the case $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$.

We recall that if $A : \mathcal{H}_1 \to \mathcal{H}_2$ is continuous, then the pullback $A' : \mathcal{H}_2' \to \mathcal{H}_1'$, defined by $A'(f) = f \circ A$ is a bounded linear map if f is bounded.

If $\phi_1 : \mathcal{H}_1 \to \mathcal{H}_1'$, $\phi_2 : \mathcal{H}_2 \to \mathcal{H}_2'$ are the conjugate linear maps from Riesz Rep, we define the adjoint A^*, a map from $\mathcal{H}_2$ to $\mathcal{H}_1$:

$$A^* = \phi_1^{-1} \circ A' \circ \phi_2$$

Since ϕ_1, ϕ_2 are conjugate linear isometries, A^* is linear, and we have the following proposition:

1.43 Proposition. $\|A\| = \|A'\| = \|A^*\|$.

Proof. We use the fact in the second equality that when $\|Ax\| \neq 0$, there exists a vector $y = \frac{Ax}{\|Ax\|} \in \mathcal{H}_2$ with norm 1, so by Cauchy Schwarz,

$$\|A\| = \sup_{\|x\| \le 1} \|Ax\| = \sup_{\|x\| \le 1} \sup_{\|y\| \le 1} |\langle Ax, y \rangle|.$$

And as a consequence of Riesz Rep, we have the following:

$$\begin{aligned} \|A\| &= \sup_{\|x\| \le 1} \sup_{\|f\| \le 1} |f(Ax)| \\ &= \sup_{\|f\| \le 1} \sup_{\|x\| \le 1} |f(Ax)| \\ &= \sup_{\|f\| \le 1} \sup_{\|x\| \le 1} |(A'f)x| \\ &= \sup_{\|f\| \le 1} \|A'f\| \\ &= \|A'\|. \end{aligned}$$

Since ϕ_1, ϕ_2 are conjugate linear isometric bijections,

$$\begin{aligned} \|A'\| &= \sup_{\|f\| \le 1} \|A'f\| \\ &= \sup_{\|\phi_2(x)\| \le 1} \|A'(\phi_2(x))\| \\ &= \sup_{\|x\| \le 1} \|A'(\phi_2(x))\| && \text{(By Riesz Rep)} \\ &= \sup_{\|x\| \le 1} \|\phi_1^{-1}(A'(\phi_2(x)))\| \\ &= \|A^*\|. \end{aligned}$$

□

The following commutative diagram illustrates the relationship between the pullback and adjoint map:

$$\begin{array}{ccc} \mathcal{H}_2 & \xrightarrow{A^*} & \mathcal{H}_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \phi_1} \\ \mathcal{H}_2' & \xrightarrow[A']{} & \mathcal{H}_1' \end{array}$$

From the diagram, we acquire an important property of A^*:

$$\begin{aligned} \langle Ax, y \rangle &= \phi_2(y)(Ax) \\ &= A'(\phi_2(y))(x) \\ &= \phi_1(A^*(y))(x) \\ &= \langle x, A^*(y) \rangle. \end{aligned}$$

1.44 Theorem. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, then the adjoint map $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $A \mapsto A^*$ is a conjugate linear map that is an isometry with respect to the operator norm. Moreover, for $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$:*

1. *A^* is characterized by the identity $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for each $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$*
2. *$(BA)^* = A^* B^*$*
3. *$(A^*)^* = A$*
4. *$\|A^*A\| = \|AA^*\| = \|A\|^2$*

Proof. (1) We want to determine A^*y for any $y \in \mathcal{H}_2$. Since $x \mapsto \langle Ax, y \rangle$ is a bounded linear functional, by Riesz Representation Theorem, there is a $z \in \mathcal{H}_1$ such that

$$\langle Ax, y \rangle = \langle x, z \rangle = \langle x, A^* y \rangle.$$

So for each $x \in \mathcal{H}_1$,

$$\langle x, A^*y - z \rangle = 0.$$

Hence, $A^*y - z = 0$ or $A^*y = z$.

(2) From (1), we only need to consider inner products, say with $x \in \mathcal{H}_1$, $z \in \mathcal{H}_3$

$$\begin{aligned} \langle BAx, z \rangle_{\mathcal{H}_3} &= \langle Ax, B^*z \rangle_{\mathcal{H}_2} \\ &= \langle x, A^*B^*z \rangle_{\mathcal{H}_1} \end{aligned}$$

So $(BA)^* = A^*B^*$.

(3) We have for $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$,

$$\begin{aligned} \langle Ax, y \rangle &= \langle x, A^*y \rangle \\ &= \overline{\langle A^*y, x \rangle} \\ &= \overline{\langle y, (A^*)^*x \rangle} \\ &= \langle (A^*)^*x, y \rangle \end{aligned}$$

So $\langle (A - (A^*)^*)x, y \rangle = 0$, thus $A = (A^*)^*$.

(4) For $x \in \mathcal{H}_1$,

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax \rangle \\ &= \langle x, A^*Ax \rangle \\ &\leq \|x\| \|A^*Ax\| \\ &\leq \|x\| \|A^*A\| \|x\| \end{aligned}$$

Supping over norm $\|x\| \leq 1$, by Proposition 1.40,

$$\|A\|^2 \leq \sup_{\|x\| \leq 1} \|Ax\|^2 \leq \|A^*A\| \leq \|A^*\| \|A\| = \|A\|^2.$$

So equality holds throughout, and $\|A\|^2 = \|A^*\|^2 = \|A^*A\|$. Flipping A and A^* also gives $\|A\|^2 = \|AA^*\|$.

□

1.45 Corollary. *If $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, $A \mapsto A^*$ is a conjugate linear isometry.*

Now we can explore relationships in order to distinguish between different types of operators.

1.46 Definition. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces, $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

(a) A is called *unitary* if $A^*A = \mathrm{id}_{\mathcal{H}_1}$ and $AA^* = \mathrm{id}_{\mathcal{H}_2}$.

(b) If $\mathcal{H}_1 = \mathcal{H}_2$, then A is called *self-adjoint*, or *Hermitian*, if $A^* = A$.

It is called *skew-Hermitian* if $A^* = -A$.

(c) If $\mathcal{H}_1 = \mathcal{H}_2$ and $A^*A = AA^*$, then we say that A is *normal*.