Operators on Hilbert Spaces

Lecture Notes from September 6, 2022

taken by Manpreet Singh

Last Time

- Orthogonal family
- Orthonormal basis

Warm up

- Describe the dual space of $l^2([0,1])$.
  Solution: Each element in $(l^2([0,1]))'$ has the following form:

$$\lambda : l^2([0,1]) \to \mathbb{C}$$

$$x \mapsto \sum_{j\in[0,1]} x_j \bar{y_j}$$

  with $y \in l^2([0,1])$.
  Now, we prepare for the discussion of spectral theory and the main ingredient for that is the map from an operator A to its adjoint A^*. Recall,the equivalence between continuity and boundedness.

1.39 Theorem. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ a linear map, then A is continuous $\iff$ $\|A\| = \sup_{\|x\|\leq 1} \|Ax\| < \infty$*

1.40 Remark. In this case,$\|A\|$ is the Lipschitz constant of the (Lipschitz) continuous map A.

1.41 Definition. We write $\mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ for the set of all bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $\mathbb{B}(\mathcal{H})$ for the case $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$.

1.42 Proposition. *If $A : \mathcal{H}_1 \to \mathcal{H}_2$ is continuous, then $A' : \mathcal{H}_2' \to \mathcal{H}_1'$, defined by*

$$A'(f) = f \circ A$$

is a bounded linear map with $\|A'\| = \|A\|$

Proof.

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} |\langle Ax, y\rangle|$$

for $x \in \mathcal{H}_1$,$y \in \mathcal{H}_2$

$$\|A\| = \sup_{\|x\|\leq 1} \sup_{\|f\|\leq 1} |f(A(x))|$$

for $x \in \mathcal{H}_1$,$f \in \mathcal{H}_2'$

$$\|A\| = \sup_{\|f\|\leq 1} \sup_{\|x\|\leq 1} |f \circ A(x)| = \sup_{\|f\|\leq 1} \|f \circ A\| = \sup_{\|f\|\leq 1} \|A'(f)\| = \|A'\|$$

□

If $\phi_1 : \mathcal{H}_1 \to \mathcal{H}_1'$ and $\phi_2 : \mathcal{H}_2 \to \mathcal{H}_2'$ are the conjugate linear maps from the Riesz representation theorem. We can define $A^* = \phi_1^{-1} \circ A' \circ \phi_2$ a map from $\mathcal{H}_2 \to \mathcal{H}_1$. Since ϕ_1 and ϕ_2 are conjugate linear isometries, A^* is linear and we have $\|A^*\| = \|A'\| = \|A\|$. We have a commutative diagram.

$$\begin{array}{ccc} \mathcal{H}_2 & \xrightarrow{A^*} & \mathcal{H}_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \phi_1} \\ \mathcal{H}_2' & \xrightarrow{A^*} & \mathcal{H}_1' \end{array}$$

. For $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$

$$\langle Ax, y\rangle = \phi_2(y)(Ax)$$
$$\langle Ax, y\rangle = A'(\phi_2(y))(x)$$
$$\langle Ax, y\rangle = \phi_1(A^*(y))(x)$$
$$\langle Ax, y\rangle = \langle x, A^*y\rangle$$

1.43 Theorem. *Let $\mathcal{H}_1$and$\mathcal{H}_2$ be Hilbert spaces, then the adjoint map $\mathbb{B}(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$, defined as $A \to A^*$ is a conjugate linear map that is an isometry with respect to operator norm. Moreover, for $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathbb{B}(\mathcal{H}_2\mathcal{H}_3)$*

1. *A^* is characterized by the identity $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for each $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$*
2. *$(BA)^* = A^*B^*$*
3. *$(A^*)^* = A$*
4. *$\|A^*A\| = \|AA^*\| = \|A\|^2$*

Proof. 1. We want to determine A^*y for any $y \in \mathcal{H}_2$. Since $x \mapsto \langle Ax, y\rangle$ is a bounded linear functional. By Riesz representation theorem, there is $z \in \mathcal{H}_1$ such that

$$\langle Ax, y\rangle = \langle x, z\rangle$$

We deduce that

$$\langle x, A^*y\rangle = \langle x, z\rangle$$

or, for each $x \in \mathcal{H}_1$,

$$\langle x, A^*y - z\rangle = 0$$

Hence $A^*y - z = 0 \implies A^*y = z$.

2. From 1, we only need to consider inner products, say with $x \in \mathcal{H}_1$, $z \in \mathcal{H}_3$

$$\langle B(Ax), z\rangle_{\mathcal{H}_3} = \langle Ax, B^*z\rangle_{\mathcal{H}_2}$$

$$\langle B(Ax), z\rangle_{\mathcal{H}_3} = \langle x, A^*B^*z\rangle_{\mathcal{H}_1}$$

   So, $(BA)^* = A^*B^*$

3. For $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$ we have,

$$\langle Ax, y\rangle = \langle x, A^*y\rangle$$
$$\langle Ax, y\rangle = \overline{\langle A^*y, x\rangle}$$
$$\langle Ax, y\rangle = \overline{\langle y, (A^*)^*x\rangle}$$
$$\langle Ax, y\rangle = \langle (A^*)^*x, y\rangle$$

   So, $A = (A^*)^*$

4. For $x \in \mathcal{H}$

$$\|Ax\|^2 = \langle Ax, Ax\rangle$$
$$\|Ax\|^2 = \langle x, A^*Ax\rangle$$

   By Cauchy-Schwarz inequality, we get

$$\|Ax\|^2 \leq \|x\|\|A^*Ax\|$$

   By operator norm, we get

$$\|Ax\|^2 \leq \|A^*A\|\|x\|^2$$

   and so, taking sup over $\|x\| \leq 1$ we get,

$$\|A\|^2 \leq \sup_{\|x\|\leq 1} \|Ax\|^2 \leq \|A^*A\| \leq \|A^*\|\|A\|$$

   Since $\|A\| = \|A^*\|$, we get

$$\|A\|^2 \leq \sup_{\|x\|\leq 1} \|Ax\|^2 \leq \|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2$$

   So equality holds throughout and

$$\|A\|^2 = \|A^*\|^2 = \|A^*A\|$$

   Flipping A,A^* gives

$$\|A\|^2 = \|AA^*\|$$

□

1.44 Corollary. *If $\mathcal{H}$ is a Hilbert Space, then $\mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$,$A \mapsto A^*$ is a conjugate linear isometry.*

Now, we explore relationships between A and A^* in order to distinguish different types of operators.

1.45 Definition. Let $\mathcal{H}_1$,$\mathcal{H}_2$ be Hilbert spaces, $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$

1. A is called unitary if $A^*A = id_{\mathcal{H}_1}$ and $AA^* = id_{\mathcal{H}_2}$.

2. If $\mathcal{H}_1 = \mathcal{H}_2$, then A is called self adjoint or Hermitian if $A^* = A$, it is called Skew Hermitian if $A^* = -A$.

3. If $\mathcal{H}_1 = \mathcal{H}_2$ and $A^*A = AA^*$ then we say that A is normal.