Math 7320 Lecture Notes from September 8, 2022

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Warm-Up: Let's give meaning to the statement "When dealing with complex matrices, the adjoint is the transpose conjugate."

We consider $\mathcal{H} = \mathbb{C}^n$. Associate with $n \times n$ complex matrix A, the map $x \to Ax$. The inner product is $< x, y >= \sum_{i=1}^{n} x_i \bar{y}_i$ We know A^* is characterized by the identity for each $x, y \in \mathbb{C}^n$, $< Ax, y >=< x, A^* y >$,

$< Ax, y >= \sum_{i=1}^{n} Ax_i \bar{y}_i$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_j \bar{y}_i$

$= \sum_{j=1}^{n} x_j \overline{\sum_{i=1}^{n} \overline{A_{ij}^T} y_i}$

By comparing with the identity we can see that $\overline{\sum_{i=1}^{n} \overline{A_{ij}^T} y_i} = \overline{(A^* y)_j}$

Now, we will study the types of operators introduced in the last class.

1 Theorem

For $A \in \mathcal{B}(\mathcal{H})$ the following holds:

1. If A is Hermitian then A is normal
2. If A is unitary then A is normal
3. The operators AA^* and A^*A are Hermitian
4. There are uniquely determined Hermitian operators $B, C \in \mathcal{B}(\mathcal{H})$ such that $A = B + iC$
5. A is uniquely determined by the sesquilinear form $b_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, $(x, y) \to < Ax, y >$
6. The following are equivalent
   (a) the sesquilinear form b_A is Hermitian
   (b) A is Hermitian
   (c) $b_A(x, x) \in \mathbb{R}$ for each $x \in \mathcal{H}$. In this case A is determined by $x \to b_A(x, x)$
7. If A is Hermitian and $< Ax, y >= 0$ for each $x \in \mathcal{H}$ then $A = 0$

2 Proofs

1. By definition of A being normal and A=A*
2. From $AA^* = id_{\mathcal{H}} = A^*A$
3. We see $(AA^*)^* = (A^*)A^* = AA^*$ and $(A^*A)^* = A^*(A^*)^* = A^*A$
4. We get $A = Bi + C$ with the choice $B = \frac{1}{2}(A + A^*), C = \frac{1}{2i}(A - A^*)$ Moreover, if $A = B' + iC'$ with B', C' Hermitian then $A^* = (B')^* + (iC')^* = B' - iC'$ and $B' = \frac{1}{2}(A + A^*), C' = \frac{1}{2i}(A - A^*)$ implies B, C are unique
5. We have $b_A(x, y) =< Ax, y >=< x, A^*y >= \Phi(A^*y)(x)$ and by the Φ being one-to-one by the Riesz Representation Theorem A^* is uniquely determined by b_A hence also A
6. We observe for $b_A(y, x) =< Ay, x >=< y, A^*x >=< A^*x, y >= b_{A^*}(x, y)$ so $A^* = A \iff \forall x, y \in \mathcal{H}, b_A(y, x) = b_{A^*}(x, y)$ If A or b_A are Hermitian, then the Polarization Identity shows that b_A and hence A can be constructed from knowing $b_A(x, x)$ and for each $x \in \mathcal{H}$ If A is Hermitian then for $x \in \mathcal{H}, b_A(x, x) = \overline{b_A(x, x)} \in \mathbb{R}$

   Conversely, if $b_A(x, x) \in \mathbb{R}$ for each $x \in \mathcal{H}$, we can write $A = B + iC$ and we have $b_C(x, x) = Im[b_A(x, x) + ib_C(x, x)] = 0$ Now using the Polarization Identity, $b_C(x, x) = 0$ for each $x, y \in \mathcal{H}$ and hence $C = 0$. Thus $A = B$ and A is Hermitian.
7. This follows from A being uniquely determined by b_A and $A = 0$ having $b_A(x, x) =< Ax, x >=)$ for each $x \in \mathcal{H}$

Now let's examine isometries, a type of operator more general than unitaries. Recall that isometries are norm preserving.

Lemma: A is a bounded linear map such that $A : \mathcal{H}_\infty \to \mathcal{H}_\in$ is an isometry if and only if $A * A = id_{\mathcal{H}_\infty}$

Proof: If A is an isometry then then for any $x \in \mathcal{H}, < A^*Ax, x >=< Ax, Ax >= |Ax|^2 = |x|^2 =< x, x >$ Using that A^*A is Hermitian and hence uniquely characterized by $x \to < A^*Ax, x >= |x|^2$ We get $A^*A = id_{\mathcal{H}_\infty}$

Conversely, if $A^*A = id_{\mathcal{H}_\infty}$ then $|Ax|^2 =< Ax, Ax >=< A^*Ax, x >= |x|^2$ so A is an isometry.

Theorem: For a bounded linear map $A : \mathcal{H}_\infty \to \mathcal{H}_\in$ the following are equivalent

1. A is unitary
2. A is onto and preserves the inner product. For $x, y \in \mathcal{H}, < Ax, Ay >=< x, y >$
3. A is a bijection and preserves the inner product
4. A is onto and an isometry