Lecture Notes from September 08, 2022

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Last time

- The adjoint of an operator,
- properties of the adjoint map $A \mapsto A^*$,
- Types of operators: unitary, self-adjoint, normal.

Warm up:

1.47 Question. Give meaning to the statement "when dealing with (complex) matrices, taking the adjoint is the transpose conjugate."

The idea is to use the characterizing principle of the adjoint, i.e. $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for $x, y \in \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space.

That is, we consider the Hilbert Space $\mathcal{H} = \mathbb{C}^n$ associated with $n \times n$ matrix A the map $x \mapsto Ax$.

Note, the inner product is

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

Observe that,

$$\begin{aligned}
\langle Ax, y\rangle &= \sum_{i=1}^{n} (Ax)_i \overline{y_i} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_j \overline{y_i} \\
&= \sum_{j=1}^{n} x_j \overline{\sum_{i=1}^{n} \overline{A^T_{ji}} y_i} \\
&= \sum_{j=1}^{n} x_j \overline{(\overline{A^T} y)_j} \\
&= \langle x, \overline{A^T} y\rangle
\end{aligned}$$

Therefore, $A^* = \overline{A^T}$.

We now study the types of operators we introduced last time.

1.48 Theorem. *For $A \in B(\mathcal{H})$, the following hold:*

1. *If A is Hermitian, then A is normal.*
2. *If A is Unitary, then A is normal.*
3. *The operators AA^* and A^*A are Hermitian.*
4. *There are uniquely determined Hermitian operators $B, C \in B(\mathcal{H})$ such the $A = B + iC$.*
5. *A is uniquely determined by the sesquilinear form*

$$\begin{aligned} b_A : \mathcal{H} \times \mathcal{H} &\mapsto \mathbb{C}, \\ (x, y) &\mapsto \langle Ax, y \rangle. \end{aligned}$$

6. *The following are equivalent:*
   (a) *The sesquilinear form b_A is Hermitian.*
   (b) *A is Hermitian.*
   (c) *$b_A(x, x) \in \mathbb{R}$ for each $x \in \mathcal{H}$. In this case, A is determined by $x \mapsto b_A(x, x)$.*
7. *If A is Hermitian, $\mathcal{H}$ is a complex Hilbert space, and $\langle Ax, x \rangle = 0$ for each $x \in \mathcal{H}$, then $A = 0$.*

Proof. (1) By definition of A being normal and $A = A^*$.
(2) Follows from $AA^* = id_{\mathcal{H}} = A^*A$.
(3) We see that

$$\begin{aligned} (AA^*)^* &= (A^*)^*A^* \\ &= AA^* \end{aligned}$$

and

$$\begin{aligned} (A^*A)^* &= A^*(A^*)^* \\ &= A^*A. \end{aligned}$$

(4) We get $A = B + iC$ with the choice

$$B = \frac{1}{2}(A + A^*), \quad C = \frac{1}{2i}(A - A^*)$$

Moreover, if $A = B' + iC'$ with B', C' Hermitian, then

$$\begin{aligned} A^* &= (B')^* + (iC')^* \\ &= B' - iC' \end{aligned}$$

so,

$$B' = \frac{1}{2}(A + A^*)$$

and

$$C' = \frac{1}{2i}(A - A^*)$$

Hence, B and C are the unique choice.
(5) We have,

$$\begin{aligned} b_A(x, y) &= \langle Ax, y\rangle \\ &= \langle x, A^* y\rangle \\ &= \phi(A^* y)(x) \end{aligned}$$

where the last line follows from the Riesz Representation Theorem. Moreover, ϕ is injective, so A^* is uniquely determined by b_A, hence also A.
(6) Let us first show that (a) is equivalent to (b). We observe that,

$$\begin{aligned} \overline{b_A(y, x)} &= \overline{\langle Ay, x\rangle} \\ &= \overline{\langle y, A^* x\rangle} \\ &= \langle A^* x, y\rangle \\ &= b_{A^*}(x, y) \end{aligned}$$

This shows that $A^* = A$ if and only if for each $x, y \in \mathcal{H}$, $\overline{b_A(y, x)} = b_A(x, y)$ which proves that (a) and (b) are equivalent.
Let us now show that (a) and (c) are equivalent. Suppose that the sesquilinear form b_A is Hermitian. It follows that $b_A(x, x) = \overline{b_A(x, x)}$ for each $x \in \mathcal{H}$. Hence, $b_A(x, x) \in \mathbb{R}$ for each $x \in \mathcal{H}$. Moreover, since b_A is a Hermitian sesquilinear form on a complex vector space, in this case $\mathcal{H}$, the *Polarization Identity*

$$b(x, y) = \frac{1}{4}(b(x + y, x + y) - b(x - y, x - y) + ib(x + iy, x + iy) - ib(x - iy, x - iy))$$

holds for all $x, y \in \mathcal{H}$. Therefore, the form is determined by knowing $b_A(v, v)$ for all $v \in \mathcal{H}$.
Conversely, suppose $b_A(x, x) \in \mathbb{R}$ for each $x \in \mathcal{H}$. We can write $A = B + iC$ where B, C are Hermitian operators, and from this we can show that,

$$b_A(x, x) = b_B(x, x) + ib_C(x, x)$$

for all $x \in \mathcal{H}$. Moreover, since B, C are Hermitian, we have that $b_B(x, x) \in \mathbb{R}$ and $b_C(x, x) \in \mathbb{R}$. Using this and our initial assumption, it follows that,

$$b_C(x, x) = \mathrm{Im}[b_B(x, x) + ib_C(x, x)] = 0.$$

Now, the polarization identity gives that $b_C(x, y) = 0$ for each $x, y \in \mathcal{H}$. Hence,

$$C = 0.$$

Thus, $A = B$ and A is Hermitian.
(7) This follows from A being uniquely determined by b_A and $A = 0$ having $b_A(x, x) = \langle Ax, x\rangle = 0$ for each $x \in \mathcal{H}$.

□

We now examine a type of operator that is more general than unitary isometries.

1.49 Lemma. *A bounded linear map* $A : \mathcal{H}_1 \mapsto \mathcal{H}_2$ *is an isometry if and only if*

$$A^*A = \mathrm{id}_{\mathcal{H}_1}$$

Proof. If A is an isometry, then for $x \in \mathcal{H}$,

$$\begin{aligned} \langle A^*Ax, x\rangle &= \langle Ax, Ax\rangle \\ &= \|Ax\|^2 \\ &= \|x\|^2 \\ &= \langle x, x\rangle \end{aligned}$$

Using that A^*A is Hermitian and hence uniquely characterized by $x \mapsto \langle A^*Ax, x\rangle = \|x\|^2$. We get

$$A^*A = \mathrm{id}_{\mathcal{H}_1}.$$

Conversely, if $A^*A = \mathrm{id}_{\mathcal{H}_1}$, then

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax\rangle \\ &= \langle A^*Ax, x\rangle \\ &= \|x\|^2 \end{aligned}$$

So, A is an isometry. □

1.50 Theorem. *For a bounded linear map,* $A : \mathcal{H}_1 \mapsto \mathcal{H}_2$ *the following are equivalent:*

1. *A is unitary.*
2. *A is onto and preserves the inner product, i.e. for* $x, y \in \mathcal{H}$, $\langle Ax, Ay\rangle = \langle x, y\rangle$.
3. *A is a bijection and preserves the inner product.*
4. *A is onto and an isometry.*